RECOVERY OF POLYHEDRAL SCATTERERS BY A SINGLE ELECTROMAGNETIC FAR-FIELD MEASUREMENT

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Abstract. We prove that a polyhedral obstacle in $\mathbb{R}^3$ consisting of finitely many polyhedra with mixed perfect electric conductor and perfect magnetic conductor boundary conditions can be uniquely determined by a single electric or magnetic far-field measurement, namely, the far-field pattern corresponding to a single incident wave. A unique novelty of our new technique for proving the uniqueness is to realize that the existence of an "unbounded" perfect plane implies certain symmetries of the underlying scatterer.

Mathematics Subject Classification (2000). Primary. 78A46, 35R30. Secondary. 35P25, 35J05

Keywords. Inverse electromagnetic scattering, identifiability and uniqueness, polyhedral obstacle, mixed boundary conditions.

1. Introduction

The main goal of this work is to establish the global uniqueness in determining a general polyhedral obstacle together with its boundary conditions by either the electric far-field pattern or magnetic far-field pattern.

Let $\mathbf{D} \subset \mathbb{R}^3$ be an impenetrable obstacle that consists of finitely many disjoint bounded solid polyhedra. We consider the scattering due to the obstacle corresponding to the incident normalized time-harmonic electromagnetic plane waves,

$$\mathbf{E}^i(x) := \frac{\mathrm{i}}{k}\mathrm{curl}\ \mathrm{curl}\, p\, e^{\mathrm{i}kx\cdot d} = \mathrm{i}k(d\times p)\times d e^{\mathrm{i}kx\cdot d}, \tag{1.1}$$

$$\mathbf{H}^i(x) := \mathrm{curl}\, p\, e^{\mathrm{i}kx\cdot d} = \mathrm{i}k d\times p e^{\mathrm{i}kx\cdot d}, \tag{1.2}$$

where $\mathrm{i} = \sqrt{-1}$, and $p \in \mathbb{R}^3$, $k > 0$ and $d \in \mathbb{S}^2 := \{x \in \mathbb{R}^3; |x| = 1\}$ represents respectively polarization, wave number and direction of propagation. Then the associated forward scattering problem is described by the following time-harmonic Maxwell's equations (see [7]):

$$\mathrm{curl}\,\mathbf{E} - \mathrm{i}\,k\,\mathbf{H} = 0, \quad \mathrm{curl}\,\mathbf{H} + \mathrm{i}\,k\,\mathbf{E} = 0 \quad \text{in } \mathbf{G} := \mathbb{R}^3\backslash\bar{\mathbf{D}}, \tag{1.3}$$

$$\lim_{|x|\to\infty} (\mathbf{H}^s \times x - |x|\mathbf{E}^s) = 0, \tag{1.4}$$

where $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{H} = (H_1, H_2, H_3)$ are respectively the total electric and magnetic fields formed by the incident fields $\mathbf{E}^i(x)$, $\mathbf{H}^i(x)$ and scattered fields $\mathbf{E}^s(x)$ and $\mathbf{H}^s(x)$:

$$\mathbf{E}(x) = \mathbf{E}^i(x) + \mathbf{E}^s(x), \quad \mathbf{H}(x) = \mathbf{H}^i(x) + \mathbf{H}^s(x)\,.$$

Date: January 01, 2009.

To complete the description, we need to further impose some suitable boundary conditions on $\partial\mathbf{D}$. We are interested in the following two types of boundary conditions, namely, the perfect electric conductor (PEC) boundary condition

$$\nu\times\mathbf{E}=0 \qquad \text{on} \qquad \partial\mathbf{D},$$

and the perfect magnetic conductor (PMC) boundary condition

$$\nu\times\mathbf{H}=0 \qquad \text{on} \qquad \partial\mathbf{D},$$

where ν is the outward normal to $\partial\mathbf{D}$ directing to the exterior of $\mathbf{D}$. To appeal for a general study, we consider the mixed PEC and PMC boundary conditions. To this end, we let $\partial\mathbf{D}$ have a Lipschitz dissection $\partial\mathbf{D}=\Gamma_E\cup\Sigma\cup\Gamma_H$, where Γ_E and Γ_H are disjoint, relatively open subsets of $\partial\mathbf{D}$, having Σ as their common boundary (see [21]). Then we complement the direct system (1.3)-(1.4) with the following general mixed boundary condition

$$\nu\times\mathbf{E}=0 \quad \text{on} \quad \Gamma_E; \qquad \nu\times\mathbf{H}=0 \quad \text{on} \quad \Gamma_H. \tag{1.5}$$

For convenience, we write $\mathcal{B}[\mathbf{E},\mathbf{H}]=0$ for the mixed boundary condition (1.5).

The forward scattering system (1.3)-(1.5) has a unique solution $(\mathbf{E},\mathbf{H})\in H_{loc}(\mathrm{curl};\mathbf{G})\times H_{loc}(\mathrm{curl};\mathbf{G})$ (see [12] and [13], also [3] and [4]). The solution is regular in any neighborhood which does not meet corners and edges of $\mathbf{D}$ and the interface Σ between PEC and PMC boundary conditions. The singular behavior is only attached to the corners, edges and Σ (see [8]), hence both $\mathbf{E}$ and $\mathbf{H}$ are continuous up to points lying in the interior of the (open) faces on Γ_E and Γ_H. Moreover, the Cartesian components of $\mathbf{E}$ and $\mathbf{H}$ are analytic in $\mathbf{G}$ and the asymptotic behaviors of the radiating fields $\mathbf{E}^s$ and $\mathbf{H}^s$ are governed by (see [7])

$$\mathbf{E}^s(x;\mathbf{D},p,k,d)=\frac{e^{\mathrm{i}k|x|}}{|x|}\left\{\mathbf{E}_\infty(\hat{x};\mathbf{D},p,k,d)+\mathcal{O}(\frac{1}{|x|})\right\} \quad \text{as } |x|\to\infty, \tag{1.6}$$

$$\mathbf{H}^s(x;\mathbf{D},p,k,d)=\frac{e^{\mathrm{i}k|x|}}{|x|}\left\{\mathbf{H}_\infty(\hat{x};\mathbf{D},p,k,d)+\mathcal{O}(\frac{1}{|x|})\right\} \quad \text{as } |x|\to\infty, \tag{1.7}$$

uniformly for all $\hat{x}=x/|x|\in\mathbb{S}^2$. The functions $\mathbf{E}_\infty(\hat{x})$ and $\mathbf{H}_\infty(\hat{x})$ in (1.6) and (1.7) are called, respectively, the electric and magnetic far field patterns, and both are analytic on the unit sphere $\mathbb{S}^2$. It is noted above that $\mathbf{E}^s(x;\mathbf{D},p,k,d)$, $\mathbf{E}_\infty(\hat{x};\mathbf{D},p,k,d)$, etc. will be frequently used to specify their dependence on the polarization p, the wave number k and the incident direction d.

The inverse electromagnetic obstacle scattering is to determine the scatterer $\mathbf{D}$ by using measurement data of the corresponding electric (or equivalently, magnetic) far-field patterns. This inverse problem is of fundamental importance in exploring objects by electromagnetic waves and we refer to [7] for a detailed discussion. One of the most important issues in inverse scattering problem is the uniqueness, namely, is the correspondence between $\mathbf{E}_\infty(\hat{x};\mathbf{D})$ (or equivalently, $\mathbf{H}_\infty(\hat{x};\mathbf{D})$) and $\mathbf{D}$ one to one? The inverse problem is nonlinear and moreover, severely ill-posed in the sense of Hadamard (see, e.g. [7]). Hence, the uniqueness is of critical importance in both theory and numerics, we refer to [11] for a general discussion. The uniqueness for the inverse electromagnetic scattering problem with optimal measurement data has remained a longstanding open problem (see [6]). One can easily see that this inverse problem is formally determined with a single far-field measurement, namely, the far-field pattern corresponding to a single incident wave. Hence, one may anticipate the uniqueness by using the far field data from only one or at most a finite number of incident waves. It is the similar situation as that in the inverse acoustic obstacle scattering, where one utilizes acoustic far field patterns to identify the unknown object. Recently, significant progress has been achieved for the unique determination of polyhedral type scatterers in inverse acoustic scattering by means of a single or several incident waves (see [2, 5, 9, 10, 15, 18, 19, 20]). The proofs are based on various reflection principles for Helmholtz equation, in combination with suitably devised techniques, especially the path argument developed in [18]. The study for inverse acoustic scattering problems in this direction is nearly completed. Along this line, some novel reflection principles were derived in [16] and [17] for time-harmonic Maxwell's equations. They were then applied to establish some uniqueness results for the inverse electromagnetic scattering problems by using similar techniques as those developed for inverse acoustic scattering problems. Particularly, in [14] it is proved that a single incident electromagnetic wave is sufficient to uniquely recover a polyhedral obstacle with pure PEC or PMC boundary condition. The crucial idea for the path argument in [14] is to find an 'exit path' which has at most one intersection point with the "unbounded" perfect planes. We would like to remark that similar idea was also developed in [10], where uniqueness of determining a polyhedral acoustic sound-hard obstacle is proved with a single acoustic far-field measurement. The argument relies on an 'exit path' which avoids intersection with the "unbounded" Neumann planes. However, those techniques would not work for the more challenging case considered in this work, namely to recover a general polyhedral obstacle associated with the mixed boundary conditions (1.5) by electromagnetic scattering measurement corresponding to a single incident wave, which appears to be the only problem that remains unsolved on the uniqueness in determining a general polyhedral obstacle by a single far-field measurement. We also refer to the concluding remarks in [14] for a discussion on what challenges one shall encounter when proving uniqueness in the setting posed in this paper. In realizing that the existence an "unbounded" perfect plane implies certain symmetries of the underlying scatterer, we prove the uniqueness by developing some novel and more elaborate arguments.

The rest of the paper is organized as follows. In Section 2, we consider the perfect sets and perfect planes for the Maxwell's equations and derive some crucial properties for these mathematical concepts. Then we prove the symmetry results when there exists an "unbounded" perfect plane. Finally, the major uniqueness result is established in Section 3.

2. Perfect sets, perfect planes and symmetries for Maxwell's equations

We start with some notations and basic concepts. We denote an open ball in $\mathbb{R}^3$ with center x and radius r by $B_r(x)$, and its boundary by $S_r(x)$. Unless specified otherwise, ν shall always denote the outward normal to the concerned domain, or the normal to a two-dimensional plane in $\mathbb{R}^3$. A curve $\gamma = \gamma(t)(t \geq 0)$ is said to be regular if it is C^1-smooth and $\frac{d}{dt}\gamma(t) \neq 0$. In the sequel, we adopt the traditional arc-length parametrization for a regular curve.

Throughout the rest of the paper, we let $k > 0$, $p \in \mathbb{R}^3$ and $d \in \mathbb{S}^2$ be fixed. In order for (1.1) and (1.2) to give valid incident electric and magnetic fields, we should require that $p \nparallel d$. We denote by $\mathbf{E}(x) := \mathbf{E}(x;\mathbf{D},p,k,d)$ and $\mathbf{H}(x) := \mathbf{H}(x;\mathbf{D},p,k,d)$ the total electric and magnetic fields to (1.3)-(1.5).

Definition 2.1. *$\mathscr{P}_{\mathbf{E}}$ is called a perfect set of $\mathbf{E}$ in $\mathbf{G} := \mathbb{R}^3\backslash\bar{\mathbf{D}}$ if*

$$\mathscr{P}_{\mathbf{E}} = \{x\in\mathbf{G};\nu\times\mathbf{E}\,|_{\Pi\cap B_r(x)\cap\mathbf{G}}=0 \text{ for some } r>0 \\ \text{and plane } \Pi \text{ passing through } x\}.$$

Similarly, the perfect set $\mathscr{P}_{\mathbf{H}}$ of $\mathbf{H}$ is defined in $\mathbf{G}$.

For any $x\in\mathscr{P}_{\mathbf{E}}$, we denote by Π the plane involved in the definition of $\mathscr{P}_{\mathbf{E}}$. Furthermore, we let $\widetilde{\Pi}$ be the connected component of $\Pi\backslash\bar{\mathbf{D}}$ containing x, then by the analyticity of $\mathbf{E}$ in $\mathbf{G}$, we have $\nu\times\mathbf{E}=0$ on $\widetilde{\Pi}$ by classical continuation. In the sequel, such $\widetilde{\Pi}$ will be referred to as a *perfect plane* of $\mathbf{E}$. Similarly, the perfect planes of $\mathbf{H}$ are defined. In the following, we also use $\mathscr{P}_{\mathbf{E}}$ and $\mathscr{P}_{\mathbf{H}}$ respectively to denote the sets of perfect planes of $\mathbf{E}$ and $\mathbf{H}$ whenever there is no confusion caused. We set $\mathscr{P}=\mathscr{P}_{\mathbf{E}}\cup\mathscr{P}_{\mathbf{H}}$ and when a perfect plane is concerned, it is associated either with $\mathbf{E}$ or with $\mathbf{H}$.

A remarkable property concerning perfect planes is the so-called *reflection principle*, which is summarized in the following theorem (cf. [16] and [17] for details). Subsequently, we use R_Π to denote the reflection in $\mathbb{R}^3$ with respect to a plane Π and R'_Π the linear part of the affine map R_Π.

Theorem 2.2. *Let Ω be an open connected set in $\mathbf{G}:=\mathbb{R}^3\backslash\bar{\mathbf{D}}$ which is symmetric with respect to a plane Π, namely, $R_\Pi\Omega=\Omega$. Let $\widetilde{\Pi}$ be an open connected subset of Π such that $\widetilde{\Pi}\subset\Omega$. Then we have the following results:*

(i) *Suppose that $\widetilde{\Pi}$ lies on some perfect plane from $\mathscr{P}_{\mathbf{E}}\cup\mathscr{P}_{\mathbf{H}}$ and $\Sigma\subset\partial\Omega$ or $\Sigma\subset\Omega$ is an open subset of a plane such that $\nu_\Sigma\times\mathbf{E}=0$ (resp. $\nu_\Sigma\times\mathbf{H}=0$) on Σ. Then $\nu_{\Sigma'}\times\mathbf{E}=0$ (resp. $\nu_{\Sigma'}\times\mathbf{H}=0$) on $\Sigma':=R_\Pi\Sigma$.*

(ii) $\nu_\Pi\times\mathbf{E}|_{\widetilde{\Pi}}=0$ *(i.e., $\widetilde{\Pi}$ lies on some perfect plane from $\mathscr{P}_{\mathbf{E}}$) iff*
$$\mathbf{E}(x)+R'_\Pi(\mathbf{E}(R_\Pi(x)))=0,\quad \mathbf{H}(x)-R'_\Pi(\mathbf{H}(R_\Pi(x)))=0,\quad x\in\Omega;$$

(iii) $\nu_\Pi\times\mathbf{H}|_{\widetilde{\Pi}}=0$ *(i.e., $\widetilde{\Pi}$ lies on some perfect plane from $\mathscr{P}_{\mathbf{H}}$) iff*
$$\mathbf{E}(x)-R'_\Pi(\mathbf{E}(R_\Pi(x)))=0,\quad \mathbf{H}(x)+R'_\Pi(\mathbf{H}(R_\Pi(x)))=0,\quad x\in\Omega.$$

Next we fix a perfect plane $\widetilde{\Pi}$ for our subsequent discussion and let Π be the plane in $\mathbb{R}^3$ containing $\widetilde{\Pi}$. We further localize our investigation by fixing a point $x_0\in\widetilde{\Pi}\cap\mathbf{G}$ and take a sufficiently small ball $B_{\tau_0}(x_0)\subset\mathbf{G}$. $B_{\tau_0}(x_0)$ is divided by $\widetilde{\Pi}$ into two half balls, which we respectively denote by B^+ and B^-. Let $\mathbf{G}^\pm$ be respectively the connected components of $\mathbf{G}\backslash\widetilde{\Pi}$ containing $\mathrm{B}^\pm$, and $\mathbf{\Lambda}^\pm$ respectively the connected components of $\mathbf{G}^\pm\cap R_\Pi(\mathbf{G}^\mp)$ containing $\mathrm{B}^\pm$. Finally, set $\mathbf{\Lambda}=\mathbf{\Lambda}^+\cup\widetilde{\Pi}\cup\mathbf{\Lambda}^-$ and we see that $\mathbf{\Lambda}$ is a polyhedral domain which is symmetric with respect to Π and $B_{\tau_0}(x_0)\subset\mathbf{\Lambda}$. It is observed that $\partial\mathbf{\Lambda}\subset\partial\mathbf{D}\cup R_\Pi(\partial\mathbf{D})$. By the reflection principle (Theorem 2.2(i)), we know $\partial\mathbf{\Lambda}\subset\partial\mathbf{D}\cup\mathscr{P}$. It is also observed that the construction of $\mathbf{\Lambda}$ is irrelevant to the choice of x_0 and τ_0 and is only dependent on $\widetilde{\Pi}$. In the following, we shall always write $\mathbf{\Lambda}_{\widetilde{\Pi}}$ to denote the symmetric set constructed as above corresponding to some perfect plane $\widetilde{\Pi}$. Now, we set

$$\mathscr{P}_1=\{\widetilde{\Pi};\ \widetilde{\Pi}\text{ is a perfect plane with bounded }\mathbf{\Lambda}_{\widetilde{\Pi}}\},\tag{2.1}$$

$$\mathscr{P}_2=\{\widetilde{\Pi};\ \widetilde{\Pi}\text{ is a perfect plane with unbounded }\mathbf{\Lambda}_{\widetilde{\Pi}}\}.\tag{2.2}$$

It is remarked that one always has $\widetilde{\Pi} \in \mathscr{P}_2$ if $\widetilde{\Pi}$ is an unbounded perfect plane. In fact, in such case one can verify directly that the corresponding $\mathbf{\Lambda}_{\widetilde{\Pi}}$ would contain the exterior of a sufficiently large ball containing $\mathbf{D}$. On the other hand, if $\widetilde{\Pi} \in \mathscr{P}_2$ is bounded, $\mathbf{\Lambda}_{\widetilde{\Pi}}$ would contain the exterior of a sufficiently large ball, say B_0, by noting that $\partial\mathbf{\Lambda}$ is bounded. By the reflection principle of Theorem 2.2 (ii) & (iii), $\nu \times \mathbf{E} = 0$ or $\nu \times \mathbf{H} = 0$ on $\Pi\backslash \mathrm{B}_0$ depending on whether $\nu \times \mathbf{E} = 0$ or $\nu \times \mathbf{H} = 0$ on $\widetilde{\Pi}$. That is, a bounded $\widetilde{\Pi} \in \mathscr{P}_2$ implies the existence of some unbounded perfect planes which are coplanar to $\widetilde{\Pi}$, and in this sense, it is essentially "unbounded". Denote

$$\mathscr{P}_{2,\mathbf{E}} = \mathscr{P}_2 \cap \mathscr{P}_{\mathbf{E}} \qquad \text{and} \qquad \mathscr{P}_{2,\mathbf{H}} = \mathscr{P}_2 \cap \mathscr{P}_{\mathbf{H}}.$$

In the subsequent discussion and throughout the rest of the paper, we denote by $\widetilde{\Pi}_l$, with l being an integer, a perfect plane, and Π_l the entire plane in $\mathbb{R}^3$ containing $\widetilde{\Pi}_l$. As we did earlier, we can construct a symmetric domain $\mathbf{\Lambda}_{\widetilde{\Pi}_l}$ associated with the perfect plane $\widetilde{\Pi}_l$.

Lemma 2.3. *If $\mathscr{P}_{2,\mathbf{E}}$(resp. $\mathscr{P}_{2,\mathbf{H}}$) is not empty, then there exists a plane $\Pi_{\mathbf{E}}$(resp. $\Pi_{\mathbf{H}}$) in $\mathbf{R}^3$ such that $\mathscr{P}_{2,\mathbf{E}} \subset \Pi_{\mathbf{E}}$(resp. $\mathscr{P}_{2,\mathbf{H}} \subset \Pi_{\mathbf{H}}$). Moreover if both $\mathscr{P}_{2,\mathbf{E}}$ and $\mathscr{P}_{2,\mathbf{H}}$ are not empty, we have $\Pi_{\mathbf{E}} \perp \Pi_{\mathbf{H}}$.*

Proof. Assume that $\mathscr{P}_{2,\mathbf{E}}$ is not empty. First, we claim that all the perfect planes in $\mathscr{P}_{2,\mathbf{E}}$ are perpendicular to the vector $(d \times p) \times d$. To see this, let $\widetilde{\Pi}$ be one such plane in $\mathscr{P}_{2,\mathbf{E}}$. By the arguments following equation (2.2), we know that $\widetilde{\Pi}$ are essentially "unbounded". Hence, without loss of generality, we may assume that $\widetilde{\Pi}$ is unbounded. By the definition of a perfect plane we know $\nu_{\widetilde{\Pi}} \times \mathbf{E}(x) = 0$ for $x \in \widetilde{\Pi}$, then one can directly verify that $\nu_{\widetilde{\Pi}} \times ((d \times p) \times d) = 0$ by noting the fact that $\mathbf{E}(x) = \mathbf{E}^i(x) + \mathbf{E}^s(x)$ and $\mathbf{E}^s(x) \to 0$ as $|x| \to \infty$, hence we have $(d \times p) \times d \perp \widetilde{\Pi}$, and our claim follows.

Similarly we can prove that all the perfect planes in $\mathscr{P}_{2,\mathbf{H}}$ are perpendicular to the vector $d \times p$.

Next, we show that all the perfect planes in $\mathscr{P}_{2,\mathbf{E}}$ lie on one plane in $\mathbb{R}^3$, which we denote by $\Pi_{\mathbf{E}}$. Assume contrarily that there are two perfect planes $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ belonging to $\mathscr{P}_{2,\mathbf{E}}$ such that $\Pi_1 \neq \Pi_2$. As we did earlier, it may be assumed again that both $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ are unbounded. By applying the reflection principle in Theorem 2.2 (i) repeatedly, we can get a sequence of perfect planes $\widetilde{\Pi}_l \in \mathscr{P}_{2,\mathbf{E}}, l = 1, 2, 3, \ldots$, such that $\widetilde{\Pi}_l = R_{\Pi_{l-1}}(\widetilde{\Pi}_{l-2}) \cap \mathbf{G}, l = 3, 4, \ldots$. By the previously proved result, we know that these planes $\widetilde{\Pi}_l$ are perpendicular to the same vector $(d \times p) \times d$, so they must be all parallel to each other and equidistant. This implies the existence of a perfect plane, say $\widetilde{\Pi}_{l_0}$, from this sequence such that the scatterer $\mathbf{D}$ lies entirely on one side of Π_{l_0}. Then by the definition of a perfect plane and the fact that $\Pi_{l_0} \subset \mathbf{G}$, we know $\widetilde{\Pi}_{l_0} = \Pi_{l_0}$. Now, using the reflection principle again, we know that all the faces of $R_{\Pi_{l_0}}(\mathbf{D})$ are either in $\mathscr{P}_{2,\mathbf{E}}$ or in $\mathscr{P}_{2,\mathbf{H}}$, so they are perpendicular to the vector $d \times p$ or the vector $(d \times p) \times d$ by the results proved in the first part. This is impossible since $R_{\Pi_{l_0}}(\mathbf{D})$ is the reflection of the scatterer $\mathbf{D}$, which consists of finitely many solid polyhedra.

The case when $\mathscr{P}_{2,\mathbf{H}}$ is not empty can be treated exactly in the same manner as for $\mathscr{P}_{2,\mathbf{E}}$ above.

The result $\Pi_{\mathbf{E}} \perp \Pi_{\mathbf{H}}$ follows from the facts that $\Pi_{\mathbf{E}} \perp ((d\times p)\times d)$ and $\Pi_{\mathbf{H}} \perp (d\times p)$. □

Lemma 2.4. *The open sets* $\mathbf{G}\backslash\bar{\mathscr{P}}_{2,\mathbf{E}}$ *and* $\mathbf{G}\backslash\bar{\mathscr{P}}_{2,\mathbf{H}}$ *have no bounded connected components.*

Proof. By contradiction, assume that $\mathbf{G}_1$ is a bounded connected component of $\mathbf{G}\backslash\bar{\mathscr{P}}_{2,\mathbf{E}}$, then $\partial\mathbf{G}_1 \subset \partial\mathbf{D}\bigcup\mathscr{P}_{2,\mathbf{E}}$. Now we claim that $\partial\mathbf{G}_1 \not\subset \partial\mathbf{D} = \partial\mathbf{G}$, otherwise $\partial\mathbf{G}_1 \subset \partial\mathbf{G}$ which will lead to a contradiction that $\mathbf{G}_1 = \mathbf{G}$. If $\mathbf{G}_1 \neq \mathbf{G}$, we may take one point $x \in \mathbf{G}\setminus\mathbf{G}_1$, and another point $\tilde{x} \in \mathbf{G}_1 \subset \mathbf{G}$, then we can find a path lying completely in $\mathbf{G}$ that connects x and $\tilde{x}$ by the connectedness of $\mathbf{G}$. Noting the boundedness of $\mathbf{G}_1$, the path has an intersection point with $\partial\mathbf{G}_1$, and clearly the intersection point lies in $\partial\mathbf{G}_1$ but not in $\partial\mathbf{G}$, contradicting to the relation that $\partial\mathbf{G}_1 \subset \partial\mathbf{G}$.

Clearly the above claim and the relation that $\partial\mathbf{G}_1 \subset \partial\mathbf{D}\bigcup\mathscr{P}_{2,\mathbf{E}}$ indicates the existence of an open face $\mathbf{F}$ of $\partial\mathbf{G}_1$ such that $\mathbf{F}$ lies on a perfect plane from $\mathscr{P}_{2,\mathbf{E}}$, say $\widetilde{\Pi}_0$. Associated with $\widetilde{\Pi}_0$, we can construct a symmetric domain $\mathbf{\Lambda}_{\widetilde{\Pi}_0}$; see the construction in the paragraph right after Theorem 2.2. We have $\mathbf{\Lambda}_{\widetilde{\Pi}_0} \subset \overline{\mathbf{G}_1 \cup R_{\Pi_0}\mathbf{G}_1}$ by its construction. Since $\mathbf{G}_1$ is bounded by the assumption, so is $\mathbf{\Lambda}_{\widetilde{\Pi}_0}$. Thus by the definition of $\mathscr{P}_{1,\mathbf{E}}$ we know $\widetilde{\Pi}_0 \in \mathscr{P}_{1,\mathbf{E}}$, which contradicts to the fact that $\widetilde{\Pi}_0 \in \mathscr{P}_{2,\mathbf{E}}$. This completes the proof. □

By Lemma 2.3, it is interesting to note that, if $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$, then $\mathbf{G}\backslash\bar{\mathscr{P}}_{2,\mathbf{E}}$ has exactly two unbounded connected components, and the same holds for $\mathscr{P}_{2,\mathbf{H}}$ and $\mathbf{G}\backslash\bar{\mathscr{P}}_{2,\mathbf{H}}$.

Lemma 2.5. *If* $\mathscr{P}_1 \neq \emptyset$*, then* $\mathscr{P}_2 \neq \emptyset$*.*

Proof. We assume contrarily that $\mathscr{P}_1 \neq \emptyset$ while $\mathscr{P}_2 = \emptyset$. Let $\widetilde{\Pi}_1 \in \mathscr{P}_1$ and $\gamma(t)(t \geq 0)$ be a regular curve such that $\gamma(t_1) \in \widetilde{\Pi}_1$ with $t_1 = 0$, $\gamma(t > 0) \subset \mathbf{G}\backslash\widetilde{\Pi}_1$ and γ connects to infinity. Noting that $\mathbf{\Lambda}_{\widetilde{\Pi}_1}$ is bounded, we have $\gamma \cap \partial\mathbf{\Lambda}_{\widetilde{\Pi}_1} \neq \emptyset$. Let $x_2 = \gamma(t_2)$ be the 'last' intersection point of γ with $\partial\mathbf{\Lambda}_{\widetilde{\Pi}_1}$; namely, $t_2 = \max\{t > 0; \gamma(t) \in \partial\mathbf{\Lambda}_{\widetilde{\Pi}_1}\}$, and this implies the existence of a perfect plane $\widetilde{\Pi}_2$ passing though x_2 which is extended from an open face of $\partial\mathbf{\Lambda}_{\widetilde{\Pi}_1}$ in $\mathbf{G}$. It is clear that $B_{\tau_0}(\gamma(t_1)) \subset \mathbf{G}$, where $\tau_0 := dist(\gamma, \mathbf{D})/2 > 0$. Therefore, one can verify directly that $B_{\tau_0}(\gamma(t_1)) \subset \mathbf{\Lambda}_{\widetilde{\Pi}_1}$, which then implies $|t_2 - t_1| > \tau_0$. By further noting $\mathscr{P}_2 = \emptyset$, we have $\widetilde{\Pi}_2 \in \mathscr{P}_1$. Continuing with the above arguments, we can construct a sequence of perfect planes $\widetilde{\Pi}_n, n = 2, 3, \ldots,$ and a strictly increasing sequence $t_n, n = 1, 2, \ldots$, such that $\widetilde{\Pi}_n \in \mathscr{P}_1$, $\gamma(t_n) \in \widetilde{\Pi}_n$ and $|t_{n+1} - t_n| > \tau_0$. Since $\mathscr{P}_1 \subset \overline{ch(\mathbf{D})}$, where $ch(\mathbf{D})$ is the convex hull of $\mathbf{D}$ and is obviously bounded. Hence there exits a constant $T < \infty$ such that $\lim_{n\to\infty} t_n = T$. In turn, we have $|t_{n+1} - t_n| \to 0$ as $n \to \infty$, contradicting to our construction. The proof is completed. □

Lemma 2.6. *We have* $\mathscr{P}_1 = \emptyset$*.*

Proof. Clearly, we need only to show that $\mathscr{P}_1 = \emptyset$ if $\mathscr{P}_2 \neq \emptyset$ by using Lemma 2.5. By contradiction, we assume both $\mathscr{P}_1$ and $\mathscr{P}_2$ are not empty and $\widetilde{\Pi}_1 \in \mathscr{P}_1$ is a bounded perfect plane. Noting $\mathscr{P}_2 = \mathscr{P}_{2,\mathbf{E}} \cup \mathscr{P}_{2,\mathbf{H}}$, we have $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$ or $\mathscr{P}_{2,\mathbf{H}} \neq \emptyset$. We first consider the case that both $\mathscr{P}_{2,\mathbf{E}}$ and $\mathscr{P}_{2,\mathbf{H}}$ are not empty, and at the end of the proof we would indicate that the case $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$ and $\mathscr{P}_{2,\mathbf{H}} = \emptyset$ (or, $\mathscr{P}_{2,\mathbf{E}} = \emptyset$ and $\mathscr{P}_{2,\mathbf{H}} \neq \emptyset$) can be proved similarly. In the following, we let $\Pi_{\mathbf{E}}$ and $\Pi_{\mathbf{H}}$ be the two planes stated in Lemma 2.3.

We first construct a regular $\gamma(t)(t \geq 0)$ such that $x_1 := \gamma(t_1 = 0) \in \widetilde{\Pi}_1$ and the following 4 conditions are satisfied:

(i) $\gamma(t > 0) \subset \mathbf{G}$ and γ connects to infinity;

(ii) γ has at most one intersection point with $\mathscr{P}_{2,\mathbf{E}}$;

(iii) γ does not meet the common part between two planes $\Pi_{\mathbf{E}}$ and $\Pi_{\mathbf{H}}$.

(iv) $\gamma(t)$ does not meet the planes $\Pi_{\mathbf{E}}$ and $\Pi_{\mathbf{H}}$ for sufficiently large t.

Note that condition (iii) and (iv) can be easily satisfied by making proper small deformation of the curve, so we need only to find a curve satisfying condition (i) and (ii). This can be done as follows: First note that $\mathbf{G}$ is connected and unbounded, we can find a regular curve $\gamma_1(t)(t \geq 0)$ such that $x_1 := \gamma_1(t_1 = 0) \in \widetilde{\Pi}_1$, $\gamma_1(t > 0) \subset \mathbf{G}$ and γ_1 connects to infinity. If $\gamma_1(t)(t \geq 0)$ does not intersect $\mathscr{P}_{2,\mathbf{E}}$ more than once, then γ_1 is our desired curve. On the other hand, if it intersects $\mathscr{P}_{2,\mathbf{E}}$ more that once, let $\gamma_1(t^*) \in \Pi^* \subset \mathscr{P}_{2,\mathbf{E}}$ be the first intersection point. By Lemma 2.4, Π^* belongs to some unbounded component of $\mathbf{G}\backslash\bar{\mathscr{P}}_{2,\mathbf{E}}$, thus we can construct another regular curve $\gamma_2(t)(t \geq 0)$ such that $\gamma_2(0) = \gamma_1(t^*)$, $\gamma_2(t > 0) \subset \mathbf{G}\backslash\bar{\mathscr{P}}_{2,\mathbf{E}}$ and γ_2 connects to infinity. Now we can concatenate the part of the curve $\gamma_1(t)$ for $0 \leq t \leq t^*$ with the one $\gamma_2(t)$ $(t \geq 0)$ and modify the concatenated curve around the joining point so that the resulting curve, denoted by $\gamma(t)$, is regular. Clearly $\gamma(t)$ satisfies the requirements. In the sequel, we set $d_0 = dist(\gamma, \mathbf{D}) > 0$ and $r_0 = d_0/2$.

This regular curve γ will act as the 'exit path' for our subsequent *path argument* to prove the lemma. More specifically, we shall construct a sequence of pairs $(\gamma(t_n), \widetilde{\Pi}_n)$, $n = 2, 3, 4....$ such that $t_n < t_{n+1}$, $\widetilde{\Pi}_n \subset \mathscr{P}_1 \cup \mathscr{P}_2$ and $\gamma(t_n) \in \widetilde{\Pi}_n$. Moreover, each $\widetilde{\Pi}_n$ is extended from an open face of the boundary of a bounded domain in $\mathbf{G}$ whose boundary belongs to $\partial\mathbf{D} \cup \mathscr{P}_1 \cup \mathscr{P}_2$. Using this sequence a contradiction can be derived. We carry out the construction of the sequence now by induction.

We first construct $(\gamma(t_2), \widetilde{\Pi}_2)$. Since $\widetilde{\Pi}_1 \in \mathscr{P}_1$, we know that the corresponding symmetric set $\mathbf{\Lambda}_{\widetilde{\Pi}_1}$ is bounded and hence $\gamma \cap \partial\mathbf{\Lambda}_{\widetilde{\Pi}_1} \neq \emptyset$. Let $x_2 = \gamma(t_2)$ be the 'last' intersection point of γ with $\partial\mathbf{\Lambda}_{\widetilde{\Pi}_1}$, and this then implies the existence of a perfect plane $\widetilde{\Pi}_2$ passing though x_2 which is extended from an open face of $\partial\mathbf{\Lambda}_{\widetilde{\Pi}_1}$ in $\mathbf{G}$. Furthermore, it is clear that $|t_2 - t_1| > r_0$. So we have determined $(\gamma(t_2), \widetilde{\Pi}_2)$.

Next, we assume that we have constructed the pair $(\gamma(t_n), \widetilde{\Pi}_n)$ for $n \geq 2$ such that $\widetilde{\Pi}_n \subset \mathscr{P}_1 \cup \mathscr{P}_2$, $\gamma(t_n) \in \widetilde{\Pi}_n$ and $\widetilde{\Pi}_n$ is extended from an open face of the boundary of a bounded domain in $\mathbf{G}$, denoted by $\mathbf{\Upsilon}_n$, whose boundary belongs to $\partial\mathbf{D} \cup \mathscr{P}_1 \cup \mathscr{P}_2$. We proceed to construct $(\gamma(t_{n+1}), \widetilde{\Pi}_{n+1})$. Obviously we have either $\widetilde{\Pi}_n \in \mathscr{P}_1$ or $\widetilde{\Pi}_n \in \mathscr{P}_2$. If $\widetilde{\Pi}_n \in \mathscr{P}_1$, we repeat the above argument to find a perfect plane $\widetilde{\Pi}_{n+1}$ which is extended from some open face of $\partial\mathbf{\Lambda}_{\widetilde{\Pi}_n}$ and $x_{n+1} := \gamma(t_{n+1}) \in \widetilde{\Pi}_{n+1}$ such that $|t_{n+1} - t_n| > r_0$. On the other hand, if $\widetilde{\Pi}_n \in \mathscr{P}_n$ we have either $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{E}}$ or $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{H}}$. Then we can show

(1) If $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{E}}$, then there exists either an $x_{n+1} := \gamma(t_{n+1}) \in \widetilde{\Pi}_{n+1} \in \mathscr{P}_1 \cup \mathscr{P}_{2,\mathbf{H}}$ such that $|t_{n+1} - t_n| > 0$, or an $x_{n+1} := \gamma(t_{n+1}) \in \widetilde{\Pi}_{n+1} \in \mathscr{P}_1 \cup \mathscr{P}_2$ such that $|t_{n+1} - t_n| > r_0$. In both cases, $\widetilde{\Pi}_{n+1}$ is extended from an open face of the boundary of a bounded domain in $\mathbf{G}$ whose boundary belongs to $\partial\mathbf{D} \cup \mathscr{P}_1 \cup \mathscr{P}_2$.

(2) If $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{H}}$, then the same result as in (1) holds with "$\mathscr{P}_{2,\mathbf{H}}$" being replaced by $\mathscr{P}_{2,\mathbf{E}}$.

We argue first for the case (1) of $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{E}}$: Noting $\widetilde{\Pi}_n$ is extended from an open face, say $\mathbf{F}_n$, of the bounded domain $\mathbf{\Upsilon}_n$ in $\mathbf{G}$ whose boundary belongs to $\partial\mathbf{D} \cup \mathscr{P}_1 \cup \mathscr{P}_2$, we let Θ_0 be the connected component of $(\mathbf{\Upsilon}_n \cup \widetilde{\Pi}_n \cup R_{\Pi_n}(\mathbf{\Upsilon}_n)) \cap \mathbf{\Lambda}_{\widetilde{\Pi}_n}$ containing $\mathbf{F}_n$. Since $\mathbf{\Upsilon}_n$ is bounded, we know Θ_0 is bounded. Then by the reflection principle, we get $\partial\Theta_0 \subset \partial\mathbf{D} \cup \mathscr{P}_1 \cup \mathscr{P}_2$. Let us further specify two cases: (i) $x_n \in \Theta_0$; (ii) $x_n \in \partial\Theta_0$.

In case (i), we define $x_{n+1} := \gamma(t_{n+1})$ to be the 'last' intersection point of γ with $\partial\Theta_0$. Clearly we have a perfect plane $\widetilde{\Pi}_{n+1}$ which is extended from an open face of $\partial\Theta_0$ in $\mathbf{G}$. It is remarked that we would only have $|t_{n+1} - t_n| > 0$ but not necessarily have $|t_{n+1} - t_n| > r_0$ in this case. Noting $\widetilde{\Pi}_n$ lies on Π_n, which is exactly $\Pi_{\mathbf{E}}$ in this case, and Θ_0 is symmetric with respect to Π_n, we have $\widetilde{\Pi}_{n+1} \not\subset \Pi_{\mathbf{E}}$. Then by Lemma 2.3, we have that $\widetilde{\Pi}_{n+1} \in \mathscr{P}_1$ or $\widetilde{\Pi}_{n+1} \in \mathscr{P}_{2,\mathbf{H}}$.

In case (ii) with $x_n \in \partial\Theta_0$, we know there is some perfect plane $\widetilde{\Pi}'_n$ other than $\widetilde{\Pi}_n$ which contains x_n and is extended from a face of $\partial\Theta_0$. Using Lemma 2.3, we see that $\widetilde{\Pi}'_n \not\subset \mathscr{P}_{2,\mathbf{E}}$. Besides it holds that $\widetilde{\Pi}'_n \not\subset \mathscr{P}_{2,\mathbf{H}}$, otherwise $x_n \in \widetilde{\Pi}'_n \cap \widetilde{\Pi}_n \subset \Pi_{\mathbf{E}} \cap \Pi_{\mathbf{H}} = L$, contradicting to our construction of γ. Hence we have $\widetilde{\Pi}'_n \in \mathscr{P}_1$. Using this fact, we can easily see that $x_{n+1} := \gamma(t_{n+1})$, the 'last' intersection point between γ and $\mathbf{\Lambda}_{\widetilde{\Pi}'_n}$, satisfies $|t_{n+1} - t_n| > r_0$, and the existence of a perfect plane $\widetilde{\Pi}_{n+1} \in \mathscr{P}_1 \cup \mathscr{P}_2$ passing through x_{n+1}.

The argument for the case (2) with $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{H}}$ is exactly the same as for the case (1) with $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{E}}$.

In the above we have constructed the pair $(\gamma(t_{n+1}), \widetilde{\Pi}_{n+1})$ in every possible cases. It is clear from our construction that $\widetilde{\Pi}_{n+1}$ is extended from a face of the boundary of a bounded domain in $\mathbf{G}$ whose boundary belongs to $\partial\mathbf{D} \cup \mathscr{P}_1 \cup \mathscr{P}_2$.

Now, we have finished the construction of the desired sequence of pairs $(\gamma(t_n), \widetilde{\Pi}_n)$, $n = 2, 3, 4, ...$ We claim that $\gamma(t_n) \to \infty$ as $n \to \infty$. Indeed, according to our construction of $\gamma(t)$, we know that it has at most one intersection point with $\mathscr{P}_{2,\mathbf{E}}$. So we may assume that $\widetilde{\Pi}_{n_0} \in \mathscr{P}_{2,\mathbf{E}}$ for some integer n_0, where it may happen that $n_0 = 0$, i.e., all the $\gamma(t_n)$'s with $n = 1, 2, \ldots$ do not belong to $\mathscr{P}_{2,\mathbf{E}}$. Then for all $n > n_0$, we have $\widetilde{\Pi}_n \in \mathscr{P}_1 \cup \mathscr{P}_{2,\mathbf{H}}$. i.e. either $\widetilde{\Pi}_n \in \mathscr{P}_1$ or $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{H}}$. In the former case, we have $|t_{n+1} - t_n| > r_0$ from the previous construction. While in the latter case, we know from case (2) above that either $|t_{n+1} - t_n| > r_0$ or $|t_{n+1} - t_n| > 0$ with $\gamma(t_{n+1}) \in \widetilde{\Pi}_{n+1} \subset \mathscr{P}_1$, in that case we can apply the same argument as in the former case to show that $|t_{n+2} - t_{n+1}| > r_0$. Thus we can conclude $|t_{n+2} - t_n| > r_0$ for all $n > n_0$, from which our claim follows immediately.

Next, by the construction of γ and Lemma 2.3, we see that $\gamma(t)$ does not intersect $\mathscr{P}_2 = \mathscr{P}_{2,\mathbf{E}} \cup \mathscr{P}_{2,\mathbf{H}}$ for sufficiently large t. Therefore $\gamma(t_n) \in \mathscr{P}_1$ for all n sufficiently large. But from the definition of $\mathscr{P}_1$, we clearly have $\mathscr{P}_1 \subset \overline{ch(\mathbf{D})}$, which is bounded. This contradicts to our construction that $t_n \to \infty$ as $n \to \infty$, thus completes the proof of the case when both $\mathscr{P}_{2,\mathbf{E}}$ and $\mathscr{P}_{2,\mathbf{H}}$ are not empty.

Finally we come to the remaining two cases: (a) $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$ and $\mathscr{P}_{2,\mathbf{H}} = \emptyset$; and (b) $\mathscr{P}_{2,\mathbf{H}} \neq \emptyset$ and $\mathscr{P}_{2,\mathbf{E}} = \emptyset$. For the case (a), we apply the same arguments as in the case when both $\mathscr{P}_{2,\mathbf{E}}$ and $\mathscr{P}_{2,\mathbf{H}}$ are not empty. It is remarked that the subcase when the perfect plane $\widetilde{\Pi}_n$'s belong to $\mathscr{P}_{2,\mathbf{H}}$ can not happen now in the construction of the pair $(\gamma(t_{n+1}), \widetilde{\Pi}_{n+1})$ since we have $\mathscr{P}_{2,\mathbf{H}} = \emptyset$. This fact leads to the same contradiction, but with much easier deductions. Similar arguments also hold for the case (b). □

We are in a position to present the symmetry properties of the scatterer $\mathbf{D}$.

Lemma 2.7. *If $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$, then $\mathbf{D}$ is symmetric with respect to $\Pi_{\mathbf{E}}$. Similarly, if $\mathscr{P}_{2,\mathbf{H}} \neq \emptyset$, then $\mathbf{D}$ is symmetric with respect to $\Pi_{\mathbf{H}}$. Here $\Pi_{\mathbf{E}}$ and $\Pi_{\mathbf{H}}$ are the two planes introduced in Lemma 2.3.*

Proof. We consider only the case $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$, and the other case can be treated similarly. Let $\mathbf{G}^*$ be the unique unbounded open connected component of $\mathbf{G} \cap R_{\Pi_{\mathbf{E}}}(\mathbf{G})$ and set $\mathbf{D}^* := \mathbb{R}^3 \backslash \overline{\mathbf{G}^*}$. We see that both $\mathbf{G}^*$ and $\mathbf{D}^*$ are symmetric with respect to $\Pi_{\mathbf{E}}$. It is also clear that $\mathbf{G}^* \subset \mathbf{G}$ and $\mathbf{D}^*$ is bounded. Moreover, the reflection principle in Theorem 2.2 (i) implies that $\partial \mathbf{G}^* \subset \partial \mathbf{G} \cup (\mathscr{P}_1 \cup \mathscr{P}_2)$. Clearly, if $\mathbf{G} \backslash \overline{\mathbf{G}^*} = \emptyset$, or equivalently $\mathbf{D}^* \backslash \bar{\mathbf{D}} = \emptyset$, then one has $\mathbf{D} = \mathbf{D}^*$ and the lemma follows immediately. Thus we need only to show that $\mathbf{G} \backslash \overline{\mathbf{G}^*} = \emptyset$. By contradiction, assume that $\mathbf{G} \backslash \overline{\mathbf{G}^*} \neq \emptyset$ or equivalently $\mathbf{D}^* \backslash \bar{\mathbf{D}} \neq \emptyset$. Then we can find some nonempty open connected component of $\mathbf{D}^* \backslash \bar{\mathbf{D}}$, which we denote by $\mathbf{D}^{**}$. We see that $\partial \mathbf{D}^{**} \subset \partial \mathbf{D}^* \cup \partial \mathbf{D} = \partial \mathbf{G}^* \cup \partial \mathbf{G} \subset \partial \mathbf{G} \cup (\mathscr{P}_1 \cup \mathscr{P}_2)$.

Now we have two cases: $\mathbf{D}^{**} \cap \mathscr{P}_{2,\mathbf{E}} = \emptyset$ and $\mathbf{D}^{**} \cap \mathscr{P}_{2,\mathbf{E}} \neq \emptyset$. In the first case, using Lemma 2.4 we see that $\mathbf{D}^{**}$ lies entirely in one unbounded connected component of $\mathbf{G} \backslash \bar{\mathscr{P}}_{2,\mathbf{E}}$, which we denote by $\mathbf{W}$. Whereas in the latter case when $\mathbf{D}^{**} \cap \mathscr{P}_{2,\mathbf{E}} \neq \emptyset$, we take $x_0 \in \mathbf{D}^{**} \cap \mathscr{P}_{2,\mathbf{E}}$ to be an arbitrary point, and it also follows from Lemma 2.4 that x_0 belongs to the boundary of some unbounded connected component of $\mathbf{G} \backslash \bar{\mathscr{P}}_{2,\mathbf{E}}$, which we still denote by $\mathbf{W}$. Let $\gamma(t) (t \geq 0)$ be a regular curve such that:

i) $\gamma(0) \in \mathbf{D}^{**}$, $\gamma(t > 0) \subset \mathbf{W}$ and γ connects to infinity in the case $\mathbf{D}^{**} \cap \mathscr{P}_{2,\mathbf{E}} = \emptyset$;
ii) $\gamma(0) = x_0$, $\gamma(t > 0) \subset \mathbf{W}$ and γ connects to infinity in the case $\mathbf{D}^{**} \cap \mathscr{P}_{2,\mathbf{E}} \neq \emptyset$.

By the fundamental property of a connected set, we know that $\gamma \cap \partial \mathbf{D}^{**} \neq \emptyset$ for both cases. Let $\gamma(t_1)$ with $t_1 > 0$ be the 'last' intersection point between γ and $\partial \mathbf{D}^{**}$. By analytical continuation, this implies the existence a perfect plane $\widetilde{\Pi}_1$ extended from an open part of $\partial \mathbf{D}^{**} \backslash \partial \mathbf{D}$ in $\mathbf{G}$ such that $\gamma(t_1) \in \widetilde{\Pi}_1$.

We claim that $\gamma(t_1)$ lies in the interior of some face of $\partial \mathbf{D}^{**}$, which is contained in $\widetilde{\Pi}_1$ and that $\widetilde{\Pi}_1 \subset \mathscr{P}_{2,\mathbf{H}}$. Indeed, according to our previous construction of $\gamma(t)$, we know $\gamma(t) \cap \mathscr{P}_{2,\mathbf{E}} = \emptyset$ for $t > 0$, and this, along with Lemma 2.6 which ensures $\mathscr{P}_1 = \emptyset$, concludes that only perfect planes in $\mathscr{P}_{2,\mathbf{H}}$ have intersection points with $\gamma(t)$. Clearly our claim follows immediately by noting that $\gamma(t_1)$ lies on the perfect plane $\widetilde{\Pi}_1$ and that all the perfect planes in $\mathscr{P}_{2,\mathbf{H}}$ lie on $\Pi_{\mathbf{H}}$ due to Lemma 2.3.

Now, let Θ be the connected component of $(\mathbf{D}^{**} \cup \widetilde{\Pi}_1 \cup R_{\Pi_1} \mathbf{D}^{**}) \cap \mathbf{\Lambda}_{\widetilde{\Pi}_1}$ containing $\widetilde{\Pi}_1$. Then we have from our previous claim that $\gamma(t_1) \in \Theta$. Using the same construction as in the proof of case (i) of assertion (2) in Lemma 2.6, one can show that there exists a $t_2 > t_1$ such that $\gamma(t_2) \in \widetilde{\Pi}_2 \subset \mathscr{P}_1 \cup \mathscr{P}_{2,\mathbf{E}}$. This contradicts to the previous result that only perfect planes in $\mathscr{P}_{2,\mathbf{H}}$ have intersection points with $\gamma(t)$, thus completes the proof of the lemma.

□

Lemma 2.8. *Let the scatterer* $\mathbf{D}$ *be associated with the mixed boundary condition (1.5), then it holds that* $R_{\Pi_{\mathbf{E}}}(\Gamma_{\mathbf{E}})=\Gamma_{\mathbf{E}}$ *and* $R_{\Pi_{\mathbf{E}}}(\Gamma_{\mathbf{H}})=\Gamma_{\mathbf{H}}$ *if* $\mathscr{P}_{2,\mathbf{E}}\neq\emptyset$*; and* $R_{\Pi_{\mathbf{H}}}(\Gamma_{\mathbf{E}})=\Gamma_{\mathbf{E}}$ *and* $R_{\Pi_{\mathbf{H}}}(\Gamma_{\mathbf{H}})=\Gamma_{\mathbf{H}}$ *if* $\mathscr{P}_{2,\mathbf{H}}\neq\emptyset$.

Proof. It suffices to consider the case $\mathscr{P}_{2,\mathbf{E}}\neq\emptyset$. Assume contrarily that $R_{\Pi_{\mathbf{E}}}(\Gamma_{\mathbf{E}})\neq\Gamma_{\mathbf{E}}$. By Lemma 2.7, we know that $\mathbf{D}$ is symmetric with respect to $\Pi_{\mathbf{E}}$. With the help of the reflection principle in Theorem 2.2 (i), it is straightforward to show that on an open subset of $\partial\mathbf{D}$, one has both $\nu\times\mathbf{E}=0$ and $\nu\times\mathbf{H}=0$. Hence by the unique continuation (see, e.g., Lemma 3.2 in [1]), we have $\mathbf{E}=\mathbf{H}=0$ in $\mathbf{G}$. Recalling the asymptotic behavior of the scattered field $\mathbf{E}^s$ in (1.6), we derive that $\lim_{|x|\to\infty}|\mathbf{E}^i|=0$, which is not true since $|\mathbf{E}^i|$ is a non-zero constant by (1.1). Therefore we have shown $R_{\Pi_{\mathbf{E}}}(\Gamma_{\mathbf{E}})=\Gamma_{\mathbf{E}}$, which implies also $R_{\Pi_{\mathbf{E}}}(\Gamma_{\mathbf{H}})=\Gamma_{\mathbf{H}}$. □

Remark 2.9. *From Lemma 2.7 and 2.8, we know that if* $\mathscr{P}_{2,\mathbf{E}}\neq\emptyset$*, then both the domain* $\mathbf{G}=\mathbb{R}^3\backslash\bar{\mathbf{D}}$ *and the boundary condition (1.5) are symmetric with respect to the plane* $\Pi_{\mathbf{E}}$*. Appealing to Theorem 2.2, we see that the total fields* $\mathbf{E}$ *and* $\mathbf{H}$ *are also symmetric with respect to the plane* $\Pi_{\mathbf{E}}$ *in the sense that*

$$\mathbf{E}(x)+R'_{\Pi}(\mathbf{E}(R_{\Pi}(x)))=0,\ \ \mathbf{H}(x)-R'_{\Pi}(\mathbf{H}(R_{\Pi}(x)))=0,\quad x\in\mathbf{G}. \tag{2.3}$$

On the other hand, if there exists a plane, denoted by Π*, such that* $\Pi\perp(d\times p)\times d$ *and that both the obstacle* $\mathbf{D}$ *and the boundary condition (1.5) are symmetric with respect to this plane, we can also show that the symmetry relations (2.3) hold with* $\Pi_{\mathbf{E}}$ *being replaced by* Π*, by using Theorem 2.2 and the uniqueness of the forward scattering system (1.3)-(1.5). This indicates that* $\mathscr{P}_{2,\mathbf{E}}\neq\emptyset$*, and* $\mathscr{P}_{2,\mathbf{E}}\subset\Pi$ *by the reflection principle in Theorem 2.2.*

Similar symmetry results also hold for $\mathscr{P}_{2,\mathbf{H}}$*.*

3. Uniqueness in inverse obstacle scattering

As an application of the properties of perfect sets and perfect planes derived in the previous section, we are now going to present the uniqueness in determining a general polyhedral obstacle that consists of finitely many pairwise disjoint bounded polyhedra by a single far-field measurement.

Theorem 3.1. *Let* $\mathbf{D}$ *be a polyhedral scatterer associated with the mixed boundary conditions* $\mathcal{B}[\mathbf{E},\mathbf{H}]=0$ *in (1.5). Then both* $\partial\mathbf{D}$ *and* $\mathcal{B}$ *are uniquely determined by the knowledge of* $\mathbf{E}_\infty(\hat{x};p,k,d)$ *(or equivalently* $\mathbf{H}_\infty(\hat{x};p,k,d)$*) for* $\hat{x}\in\mathbb{S}^2$ *and fixed* $p\in\mathbb{R}^3$*,* $k>0$ *and* $d\in\mathbb{S}^2$ *(with* $p\times d\neq 0$*).*

Proof. Let $\widetilde{\mathbf{D}}$ be a polyhedral scatterer associated with a boundary operator $\tilde{\mathcal{B}}$. Assume that $\mathbf{D}\neq\widetilde{\mathbf{D}}$ and

$$\mathbf{E}_\infty(\hat{x};\mathbf{D},p,k,d)=\mathbf{E}_\infty(\hat{x};\widetilde{\mathbf{D}},p,k,d)\quad\text{for}\quad x\in\mathbb{S}^2. \tag{3.1}$$

Let Ω be the unique unbounded connected component of $\mathbb{R}^3\backslash(\bar{\mathbf{D}}\cup\bar{\widetilde{\mathbf{D}}})$. By Rellich's theorem (see Theorem 6.9, [7]), we infer from (3.1) that

$$\mathbf{E}(x;\mathbf{D})=\mathbf{E}(x;\widetilde{\mathbf{D}})\qquad\text{for}\ \ x\in\Omega. \tag{3.2}$$

Next, noting $\mathbf{D}\neq\widetilde{\mathbf{D}}$, we see that either $(\mathbb{R}^3\backslash\bar{\Omega})\backslash\bar{\mathbf{D}}\neq\emptyset$ or $(\mathbb{R}^3\backslash\bar{\Omega})\backslash\bar{\widetilde{\mathbf{D}}}\neq\emptyset$. Without loss of generality, we assume the former case and let $\mathbf{D}^*$ be a connected component of $(\mathbb{R}^3\backslash\bar{\Omega})\backslash\bar{\mathbf{D}}\neq\emptyset$. Clearly, $\mathbf{D}^*$ is a bounded polyhedral domain in $\mathbf{G}=\mathbb{R}^3\backslash\bar{\mathbf{D}}$ and $\mathbf{E}(x;\mathbf{D})$ is defined over D^*. Noting $\partial\mathbf{D}^*\subset\partial\Omega\cup\partial\mathbf{D}\subset\partial\mathbf{D}\cup\partial\widetilde{\mathbf{D}}$ and using (3.2), we have perfect boundary conditions on $\partial\mathbf{D}^*$. It is obvious that $\partial\mathbf{D}^*\backslash\partial\mathbf{D} \neq \emptyset$, hence there must be an open face, say Σ_0, on $\partial\mathbf{D}^*$ that can be extended in $\mathbf{G}$ to form a perfect plane $\widetilde{\Pi}_0$. By Lemma 2.6, $\widetilde{\Pi}_0 \in \mathscr{P}_2(\mathbf{D})$. Here $\mathscr{P}_2(\mathbf{D})$ and $\mathscr{P}_2(\widetilde{\mathbf{D}})$ below represent respectively the set defined in (2.2) corresponding to the scatterer $\mathbf{D}$ and $\widetilde{\mathbf{D}}$. According to our discussion prior to Lemma 2.3, $\widetilde{\Pi}_0$ is essentially "unbounded". Without loss of generality, we may assume that $\widetilde{\Pi}_0$ is unbounded. Again by using (3.2) and the previous result that $\widetilde{\Pi}_0 \in \mathscr{P}_2(\mathbf{D})$, we see $\widetilde{\Pi}_0 \in \mathscr{P}_2(\widetilde{\mathbf{D}})$. By Lemma 2.7, $\widetilde{\mathbf{D}}$ is symmetric with respect to Π_0. But this is impossible since one of the faces of $\widetilde{\mathbf{D}}$ lies on Π_0. Hence, $\mathbf{D} = \widetilde{\mathbf{D}}$. Finally, if $\mathcal{B} \neq \tilde{\mathcal{B}}$, one can show that there is an open subset of $\partial\mathbf{D} = \partial\widetilde{\mathbf{D}}$ on which both $\mathbf{E}$ and $\mathbf{H}$ assume perfect boundary conditions, leading to a same contradiction as that in the proof of Lemma 2.8.

□

Acknowledgements

The work of HL is partly supported by NSF grant, FRG DMS 0554571. The work of JZ was substantially supported by Hong Kong RGC grants (Project 404105 and Project 404606). The authors would like to thank the anonymous referee for constructive and insightful comments which have led to a significant improvement of the results and presentation of the current work.

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