

Operators and Matrices

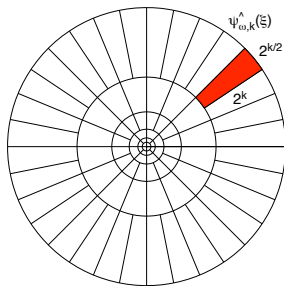
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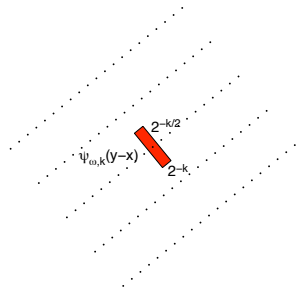
Seismic Imaging Summer School
University of Washington, 2009

Curvelets: $\varphi_\gamma(y) = 2^{-3k/4} \psi_{\omega,k}(y-x)$, $\gamma = (x, \omega, k)$

Frequency support:



Spatial support:



Distance function: $(x, \omega), (x', \omega')$, where $|\omega| = |\omega'| = 1$

Definition

$$d(x, \omega; x', \omega') = |\langle \omega, x - x' \rangle| + |\omega - \omega'|^2 + |x - x'|^2$$

$d(\cdot, \cdot)$ is an approximate distance function

- $d(x, \omega; x', \omega') \approx d(x', \omega'; x, \omega)$
- $d(x, \omega; x'', \omega'') \lesssim d(x, \omega; x', \omega') + d(x', \omega'; x'', \omega'')$

The “support” of a curvelet $\psi_{\omega, k}(y - x)$ is the set of (y, ξ)

$$d(x, \omega; y, \frac{\xi}{|\xi|}) \lesssim 2^{-k} \quad |\xi| \approx 2^k$$

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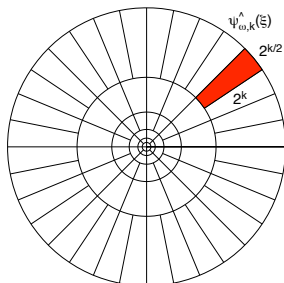
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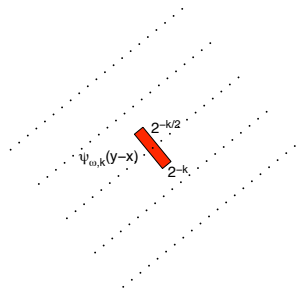
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$$\left| \omega - \frac{\xi}{|\xi|} \right|^2 \leq 2^{-k}$$

$$|\xi| \approx 2^k$$



$$|\langle \omega, y - x \rangle| + |y - x|^2 \leq 2^{-k}$$

Invariance under canonical transformations

Definition

A one-one map $\chi : (y, \eta) \rightarrow (x, \xi)$ is a homogeneous canonical transformation if

- $\chi : (y, s\eta) \rightarrow (x, s\xi), \quad s > 0$
- $\sum_i \xi_i dx_i = \sum_j \eta_j dy_j$

Remark: χ preserves phase space volume $dy d\eta$

Fact: The distance function $d(\cdot, \cdot)$ is invariant under χ

$$d(\chi(y, \omega); \chi(y', \omega')) \approx d(y, \omega; y', \omega')$$

Proof: $\langle \xi, x - x' \rangle = \langle \eta, y - y' \rangle + \text{quadratic}$

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Control on curvelet frame non-orthogonality

- Two curvelets at same scale $\gamma = (x, \omega, k)$, $\gamma' = (x', \omega', k)$

$$\int \overline{\varphi_\gamma(y)} \varphi_{\gamma'}(y) dy \lesssim [1 + 2^k d(x, \omega; x', \omega')]^{-N} \quad \forall N$$

- There are $\approx R^2$ coefficients $\gamma' = (x', \omega', k)$ with

$$2^k d(x, \omega; x', \omega') \leq R$$

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Analysis and Synthesis

Curvelet transform (analysis)

$$C : L^2(\mathbb{R}^2) \rightarrow \ell_\gamma^2 \quad (Cf)_\gamma = \int \overline{\varphi_\gamma(y)} f(y) dy$$

Reconstruction (synthesis)

$$C^* : \ell_\gamma^2 \rightarrow L^2(\mathbb{R}^2) \quad C^*\{c_\gamma\} = \sum_\gamma c_\gamma \varphi_\gamma$$

- Frame condition $C^*Cf = f$
- $\Pi = CC^* : \ell_\gamma^2 \rightarrow \ell_\gamma^2$ is projection onto range of C .

Matrix $\Pi_{\gamma\gamma'} = \langle \varphi_\gamma, \varphi_{\gamma'} \rangle$ satisfies

$$\Pi_{\gamma\gamma'} \lesssim 2^{-N|k-k'|} [1 + 2^k d(x, \omega; x', \omega')]^{-N} \quad \forall N$$

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Operators to Matrices

To a linear map $T : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$, associate the matrix

$$[T]_{\gamma\gamma'} = \langle \varphi_\gamma, T\varphi_{\gamma'} \rangle$$

- $[T] = C T C^*$
- $T = C^*[T]C$

Q: Which operators have nice, sparse matrices?

T an order m pseudo-differential operator

$$[T]_{\gamma\gamma'} \lesssim 2^{mk} 2^{-N|k-k'|} [1 + 2^k d(x, \omega; x', \omega')]^{-N} \quad \forall N$$

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Wave evolution in nonhomogeneous media

$$\partial_t u(t, y) + ip(y, D_y)u(t, y) = 0, \quad u(0, y) = f(y)$$

$$p(y, D)f(y) = \int e^{i\langle y, \xi \rangle} p(y, \xi) \hat{f}(\xi) d\xi$$

- Non-isotropic acoustic equation

$$\partial_t^2 u(t, y) = \sum_{ij} g^{ij}(y) \partial_{y_i} \partial_{y_j} u(t, y) + \text{l.o.t.}$$

$$p(y, \xi) = \pm \sqrt{\sum_{ij} g^{ij}(y) \xi_i \xi_j} + \text{l.o.t.}$$

- Elastic system with simple eigenvalues

$$\partial_t^2 u_l(t, y) = \sum_{ij} a_{ijlm}(y) \partial_{y_i} \partial_{y_j} u_m(t, y) + \text{l.o.t.}$$

$$p_M(y, \xi)^2 = \text{eigenvalues of } \sum_{ij} a_{ijlm}(y) \xi_i \xi_j$$

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Lax parametrix construction

Solution operator takes form

$$u(t, y) = \int e^{i\Phi_t(y, \xi)} a_t(y, \xi) \hat{f}(\xi) d\xi$$

- $\Phi_t(y, r\xi) = r\Phi_t(y, \xi)$
- $\Phi_t(y, \xi)$ is generating function for Hamiltonian flow of p :
If (y, η) is time t image of (x, ξ) under Hamiltonian flow

$$\dot{x} = d_\xi p(x, \xi), \quad \dot{\xi} = -d_x p(x, \xi)$$

Then

$$\eta = d_y \Phi_t(y, \xi), \quad x = d_\xi \Phi_t(y, \xi)$$

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Matrix of evolution operator: $\gamma = (\mathbf{x}, \omega, \mathbf{k})$

$$W_{t\varphi_\gamma}(\mathbf{y}) = \int e^{i\Phi_t(\mathbf{y}, \xi)} \mathbf{a}_t(\mathbf{y}, \xi) \hat{\varphi}_\gamma(\xi) d\xi$$

- Linearize phase function in ξ :

$$\Phi_t(\mathbf{y}, \xi) \approx \langle d_\xi \Phi_t(\mathbf{y}, \omega), \xi \rangle = \langle F_t^{-1}(\mathbf{y}), \xi \rangle$$

F_t is the Hamiltonian flow map in co-direction ω for time t .

- To first approximation

$$W_{t\varphi_\gamma}(\mathbf{y}) \approx \int e^{i\langle F_t^{-1}(\mathbf{y}), \xi \rangle} \hat{\varphi}_\gamma(\xi) d\xi = \varphi_\gamma(F_t^{-1}(\mathbf{y}))$$

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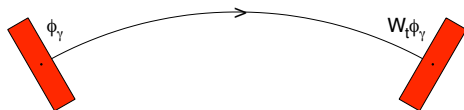
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Wave map moves a curvelet along Hamiltonian flow



More detailed analysis

$W_t \varphi_\gamma$ behaves like a curvelet at $(\chi_t(x, \omega), k)$, where χ_t is Hamiltonian flow map of p at time t .

$$|\langle \varphi_{\gamma'}, W_t \varphi_\gamma \rangle| \lesssim 2^{-N|k-k'|} [1 + 2^k d(x', \omega'; \chi_t(x, \omega))]^{-N} \quad \forall N$$

- $[W_t]_{\gamma'\gamma}$ is sparse, coefficients peak at $\gamma' = \chi_t(\gamma)$
- Matrix version of Fourier Integral Operators associated to homogeneous canonical transform χ , composition results, L^2 mapping bounds ...

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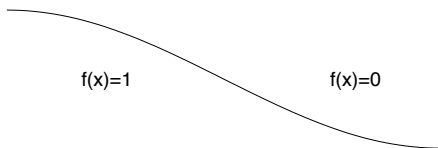
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Example: Image with sharp jump along smooth curve γ :



Wavefront set of f contained in $\Lambda = \{(\gamma(t), \xi) : \xi \perp \gamma'(t)\}$

Distance function and conormal distributions

Distributions conormal to Λ characterised by:

$$|\langle \varphi_\gamma, f \rangle| \lesssim 2^{mk} [1 + 2^k d(x, \omega; \Lambda)]^{-N} \quad \forall N$$

