# Curvelets and Wave Evolution

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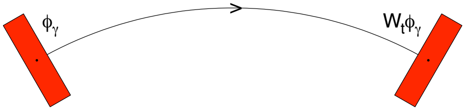
# Curvelets are "coherent" wave-packets

First approximation: curvelet follows Hamiltonian flow of $p$

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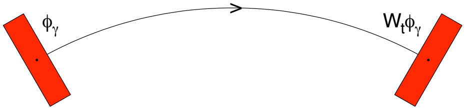


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# Accoustic half-wave equation in 2-d, $|D| = \sqrt{-\Delta_y}$

$$\partial_t u(t,y) + i\, c(y)|D|\, u(t,y) = 0\,, \qquad u(0,y) = f(y)$$

$$c(y)|D|\, f(y) = \int e^{i\langle y,\xi\rangle}\, c(y)\, |\xi|\, \hat{f}(\xi)\, d\xi$$

**Last lecture:** wrote down solution operator $W_t$, approximated.

**This lecture:** approximate equation in curvelet frame, solve.

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# Simplify action of $c(y)|D|$ on $\varphi_\gamma(y) = \psi_{\omega,k}(y-x)$

$$c(y)|D|\,\psi_{\omega,k}(y-x) = \int e^{i\langle y-x,\xi\rangle}\, c(y)\,|\xi|\,\hat{\psi}_{\omega,k}(\xi)\,d\xi$$

*Taylor expand $c(y)|\xi|$ about $(x,\omega)$, ignore any bounded error.*

- **on frequency support:** $|\xi| = \langle\omega,\xi\rangle + O(1)$
- **on spatial support:** $c(y) = c(x) + (y-x)\nabla c(x) + O(2^{-k})$

$$c(y)|D|\,\psi_{\omega,k}(y-x)$$

$$\approx \int e^{i\langle y-x,\xi\rangle}\left[\,c(x)\langle\omega,\xi\rangle + (y-x)\cdot\nabla c(x)|\xi|\,\right]\hat{\psi}_{\omega,k}(\xi)\,d\xi$$

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# Each term corresponds to an infinitesimal motion

**First term:** *infinitesimal motion of $x$ with velocity $c(x)\omega$*

$$i\int e^{i\langle y-x,\xi\rangle}c(x)\langle\omega,\xi\rangle\,\hat{\psi}_{\omega,k}(\xi)\,d\xi=-c(x)\langle\omega,\nabla_x\rangle\psi_{\omega,k}(y-x)$$

**Second term:** *infinitesimal rotation angular velocity $\nabla_{\omega^\perp}c(x)$*

$$\begin{aligned}
i\int e^{i\langle y-x,\xi\rangle}&(y-x)\cdot\nabla c(x)\,|\xi|\,\hat{\psi}_{\omega,k}(\xi)\,d\xi\\
&\approx-\int e^{i\langle y-x,\xi\rangle}\nabla c(x)\cdot|\xi|\nabla_\xi\,\hat{\psi}_{\omega,k}(\xi)\,d\xi\\
&\approx-\int e^{i\langle y-x,\xi\rangle}\nabla_{\omega^\perp}c(x)\,|\xi|\nabla_\xi^{\omega^\perp}\hat{\psi}_{\omega,k}(\xi)\,d\xi
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# Hamiltonian flow for $c(x)\,|\xi|$

$$\dot{x} = c(x)\frac{\xi}{|\xi|} \qquad \dot{\xi} = -\nabla c(x)\,|\xi|$$

Project onto co-sphere bundle, $\omega = \xi/|\xi|$

$$\dot{x} = c(x)\,\omega \qquad \dot{\omega} = -\big(\nabla_{\omega^\perp} c(x)\big)\,\omega^\perp$$

**Given $(x_t, \omega_t)$ an integral curve through $(x, \omega)$**

Let $\Theta_t$ be the rotation matrix so that $\Theta_t\omega_t = \omega$, and set

$$\varphi_\gamma(t, y) = \psi_{\omega,k}\big(\Theta_t(y - x_t)\big)$$

Then

$$\partial_t\varphi_\gamma(t, y) + ic(y)|D|\varphi_\gamma(t, y) = \tilde{\varphi}_\gamma(t, y)$$

where $\tilde{\varphi}_\gamma(t, y)$ is a “curvelet” at $(x_t, \omega_t, k)$

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# Exactly solve $\partial_t u + ic|D|u = 0\,,\ u|_0 = f$ by iteration

Let $E(t) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be defined by

$$E(t)f = \sum_\gamma c_\gamma \varphi_\gamma(t,y) \qquad \text{where} \qquad f = \sum_\gamma c_\gamma \varphi_\gamma(y)$$

Then $E(0) = I$ and $\partial_t E(t) + ic|D|E(t) = R(t)$

where the corresponding matrices $[E]$ and $[R]$ satisfy

$$[E]_{\gamma'\gamma}, [R]_{\gamma'\gamma} \lesssim 2^{N|k-k'|}\big[1 + 2^k\, d(x',\omega'; x_t,\omega_t)\big]^{-N} \quad \forall N$$

In particular, $E$ and $R$ are bounded on $L^2(\mathbb{R}^2)$.

# To solve $\partial_t u + ic|D|u = F\,,\ u|_0 = 0$

$$\text{Pose} \quad u(t,y) = \int_0^t E(t-s)G(s,y)\,ds$$

**Volterra equation on $L^2(\mathbb{R}^2)$:**

$$G(t,y) + \int_0^t R(t-s)G(s,y)\,ds = F(t,y)$$

To solve $\partial_t u + ic|D|u = 0\,,\ u|_0 = f$ , set

$$u(t) = E(t)f + v(t,y) \quad \text{where} \quad \partial_t v + ic|D|v = -R(t)f$$

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# Why bother?

This procedure works for $c(x) \in C^2(\mathbb{R}^2)$

*$C^2$ is minimal regularity for good geodesic flow.*

*Fourier integral operator approach fails.*

**Applications:**

- Bounds on size of solution $u$.
- Energy propagation results.

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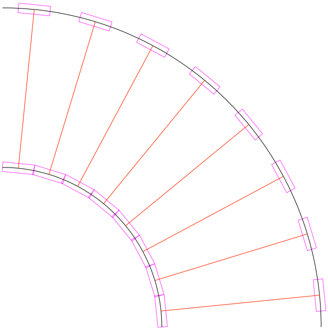
# Weakness: first approximation breaks down for $t \gtrsim 1$

Gaps develop, no curvature.



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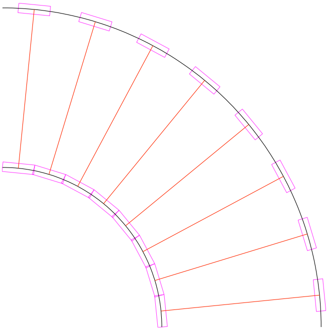

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# For better initial approximation, need quadratic terms

**First approximation:** $\exp(it|D|)\varphi_\gamma$ for $\omega = (1, 0)$:

$$\int e^{i\langle y,\xi\rangle - it|\xi|}\hat{\varphi}_\gamma(\xi)\,d\xi \approx \int e^{i\langle y,\xi\rangle - it\xi_1}\hat{\varphi}_\gamma(\xi)\,d\xi = \varphi_\gamma(y - te_1)$$

**Second approximation:** $\exp(it|D|)\varphi_\gamma$ for $\omega = (1, 0)$:

$$\int e^{i\langle y,\xi\rangle - it|\xi|}\hat{\varphi}_\gamma(\xi)\,d\xi \approx \int e^{i\langle y,\xi\rangle - it\xi_1 - it\xi_1^{-1}\xi_2^2}\hat{\varphi}_\gamma(\xi)\,d\xi$$

*Homogeneous quadratic expansion: error of size $2^{-k/2}$*

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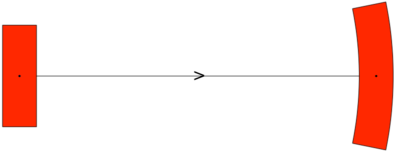
$$[exp(itD_1^{-1}D_2^2)\varphi_\gamma](y - te_1)$$



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# Second approximation captures spreading, curvature

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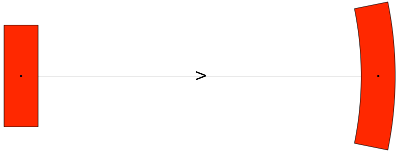

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# Parallel frame along $(x_t, \xi_t) : \partial_t \Theta_t = \Theta_t \cdot p_{\xi x}(t, x_t, \xi_t)$

# Seek $\partial_t u + ipu = O(2^{-k/2}) : u = \psi_\gamma(t, \Theta_t(y - x_t))$

$$D_t \psi_\gamma = \tfrac{1}{2}\langle A(t)\, x', x'\rangle D_1 \psi_\gamma + \tfrac{1}{2}\langle B(t)\, D', D'\rangle D_1^{-1} \psi_\gamma$$

Admits exact solution for short time:

$$\psi_\gamma(t, x) = b_t \int e^{i\langle T_t\, x, \xi\rangle + i\langle M_t\, x', x'\rangle \xi_1 + i\langle Q_t\, \xi', \xi'\rangle \xi_1^{-1}}\, \hat{\psi}_\gamma(\xi)\, d\xi$$

$$\partial_t T_t + T_t\, B(t)\, M_t = 0$$

$$\partial_t M_t + M_t\, B(t)\, M_t = -A(t)$$

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# Hamiltonian flow for $\tau + \frac{1}{2}\langle A(t)\,x', x'\rangle + \frac{1}{2}\langle B(t)\,\xi', \xi'\rangle$

- $$\partial_t\, x' = B(t)\,\xi' \qquad\qquad \partial_t\, \xi' = -A(t)\,x'$$

  $$\text{Linear}: \quad \begin{pmatrix} x' \\ \xi' \end{pmatrix} = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} \begin{pmatrix} x'_0 \\ \xi'_0 \end{pmatrix}$$

- $$T_t = W_1^{-1}, \qquad M_t = W_3\,W_1^{-1}, \qquad Q_t = -W_1^{-1}\,W_2$$

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# Conjugate point: $W_1^{-1}$ not defined

- Example: $D_t + \frac{1}{2}|x'|^2 D_1 + \frac{1}{2}|D'|^2 D_1^{-1}$

$$\text{Phase} = x_1\,\xi_1 + \sec t\,\langle x',\,\xi'\rangle - \tan t\,|x'|^2\xi_1 - \tan t\,|\xi'|^2\xi_1^{-1}$$

- $t = \frac{\pi}{2}$, Hamiltonian flow : $(x',\xi') \to (\xi'/\xi_1, -x'\xi_1)$

$$W_{\frac{\pi}{2}}\varphi_\gamma(x) \approx \frac{1+i}{\sqrt{2}}\int e^{ix_1\xi_1}\hat{\varphi}(\xi_1, -x'\xi_1)\,d\xi_1$$

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