Name:

Midterm Number 1, Math 307A/B

5 problems. No calculators or note page allowed.

The following may be useful to you.

Theorem 1 *If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation*

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

Theorem 2 *Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta, \gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem*

$$y' = f(t, y), \quad y(t_0) = y_0$$

Integrating Factors Given the following equation

$$y' + p(t)y = g(t),$$

the appropriate integrating factor is: $\mu(t) = e^{\int p(t)\, dt}$.

Euler's Method Given the following initial value problem:

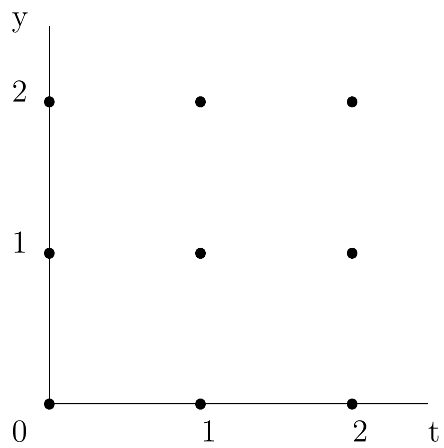
$$y' = f(t, y), \quad y(t_0) = y_0$$

Euler's method approximates a solution using:

$$y(t + h) \cong y(t) + hf(t, y)$$

1. (25 points) For the following initial value problem: $(t+1)y' + y = 1$, $y(0) = y_0$.

(i) Find the general solution in terms of y_0.
(ii) Use an appropriate theorem to tell over what interval, and for which values of y_0 the solution is guaranteed to exist. Be sure to tell why the hypothesis of the theorem are satisfied.
(iii) Draw a direction field line at every dot in the picture. Make sure it is obvious whether your lines have positive, negative, or zero slope. More detail is unnecessary.



2. (20 points) For the following initial value problem: $y' = (1 - t^2 - 2y^2)^{\frac{1}{3}}$, $y(0) = y_0$.

NOTE: Do NOT solve this equation

(i) Use an appropriate theorem to tell over what interval, and for which values of y_0 solutions are guaranteed to exist. Be sure to tell why the hypothesis of the theorem are satisfied. You may use the fact that $f(x) = x^{\frac{1}{3}}$ is continuous for all x.

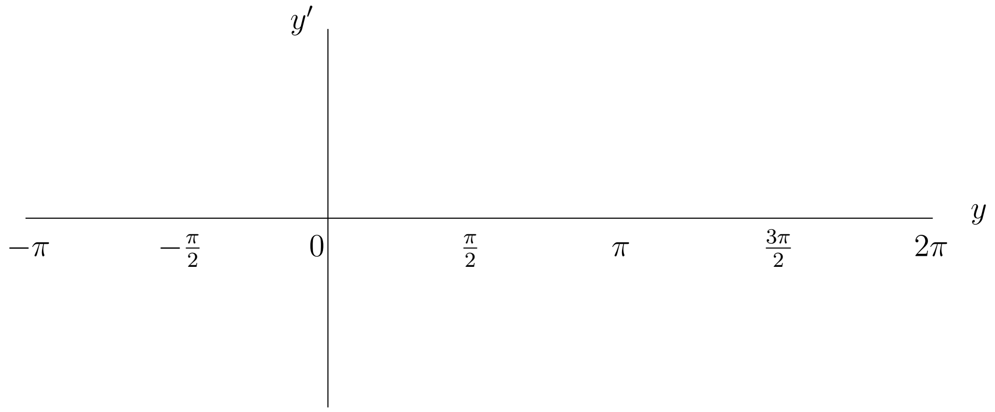
3. (15 points) For the following equation: $y' = -\sin(y)$
(i) Draw a phase diagram in the provided axes. Make sure the equilibrium are in the exact location. Don't be so exact when it comes to the other values of y'.

(ii) Use the diagram to tell the behavior of $y(t)$ as $t \to \infty$ if
a. $y(0) = 0$

b. $y(0) = \frac{\pi}{4}$

c. $y(0) = \frac{3\pi}{2}$

d. $y(0) = -\frac{\pi}{4}$



4. (20 points) For the equation $y' = y + t^2$, and starting value $y(1) = 0$, use Euler's method with a step-size of 1 to find an approximation to $y(3)$

5. (20 points) The velocity of a skydiver changes at a rate proportional to the difference between their velocity and the so called "terminal velocity" v_T, where $v_T > 0$. If the divers velocity at $t = 0$ is -1, and their velocity at $t = 5$ is -10, find an expression for their velocity at any time as a function of v_T.