Name:

Midterm Number 1, Math 307A/B

5 problems. No calculators or note page allowed.

The following may be useful to you.

Theorem 1 *If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation*

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

Theorem 2 *Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta, \gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem*

$$y' = f(t, y), \quad y(t_0) = y_0$$

Integrating Factors Given the following equation

$$y' + p(t)y = g(t),$$

the appropriate integrating factor is: $\mu(t) = e^{\int p(t)\, dt}$.

Euler's Method Given the following initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0$$

Euler's method approximates a solution using:

$$y(t + h) \cong y(t) + hf(t, y)$$

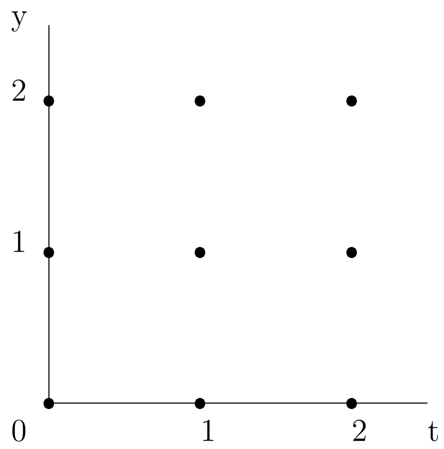
1. (25 points) For the following initial value problem: $(1+t)y' + \frac{1}{1+t}y = (1+t)e^{\frac{1}{1+t}}$, $y(0) = y_0$.

(i) Find the general solution in terms of y_0.
(ii) Use an appropriate theorem to tell over what interval, and for which values of y_0 the solution is guaranteed to exist. Be sure to tell why the hypothesis of the theorem are satisfied.
(iii) Draw a direction field line at every dot in the picture. Make sure it is obvious whether your lines have positive, negative, or zero slope. More detail is unnecessary.

SOLUTION: The integrating factor is $1+t$. Well, actually if you are careful you should get $|1+t|$. This is not differentiable (at $t=-1$). We will find out (in part (ii)) that our solution is defined only for $t > -1$ so this isn't a problem. By the way, I have been ignoring the absolute value that shows up in

$$\int \frac{1}{u}\,du = ln(|u|)$$

the reason I am able to ignore this is that the first order problems we deal with are specially made so that this is never-ever an issue.



2. (20 points) For the following initial value problem: $y' = (1 - t^2 - 2y^2)^{\frac{1}{3}}$, $y(0) = y_0$.

NOTE: Do NOT solve this equation
(i) Use an appropriate theorem to tell over what interval, and for which values of y_0 solutions are guaranteed to exist. Be sure to tell why the hypothesis of the theorem are satisfied. You may use the fact that $f(x) = x^{\frac{1}{3}}$ is continuous for all x.

SOLUTION: Here $f(t, y) = (1 - t^2 - 2y^2)^{\frac{1}{3}}$, which is continuous everywhere since it is the composition of continuous functions. $\frac{\partial f}{\partial y} = \frac{-4/3y}{(1-t^2-2y^2)^{\frac{2}{3}}}$, which is continuous except for when $1 - t^2 - 2y^2 = 0$ (everywhere else it is the composition of continuous functions). Since our starting time $t_0 = 0$, the values of y_0 that will give us problems are $y_0 = \pm\frac{1}{\sqrt{2}}$. Our conclusion is: For every $y_0 \neq \pm\frac{1}{\sqrt{2}}$, there exists a unique solution to the initial value problem defined on an interval $(-h, h)$ for some $h > 0$.

3. (15 points) For the following equation: $y' = -\sin(y)$

(i) Draw a phase diagram in the provided axes. Make sure the equilibrium are in the exact location. Don't be so exact when it comes to the other values of y'.

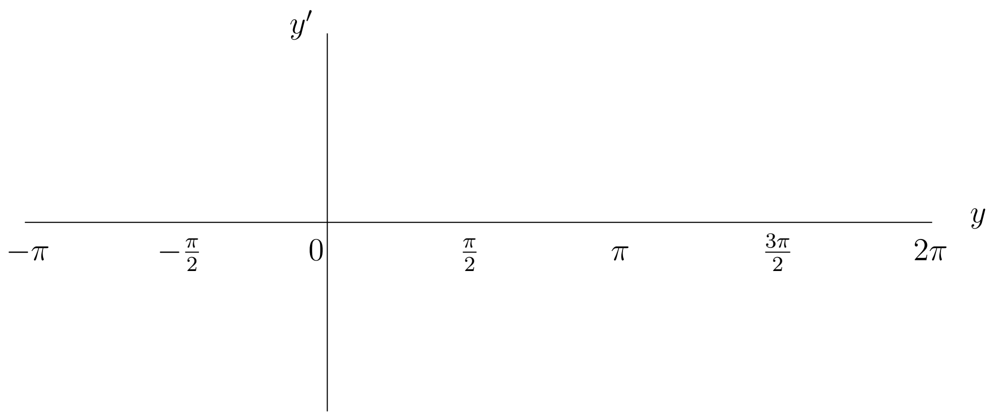
(ii) Use the diagram to tell the behavior of $y(t)$ as $t \to \infty$ if

a. $y(0) = 0$

b. $y(0) = \frac{\pi}{4}$

c. $y(0) = \frac{3\pi}{2}$

d. $y(0) = -\frac{\pi}{4}$



4. (20 points) For the equation $y' = y + t^2$, and starting value $y(1) = 0$, use Euler's method with a step-size of 1 to find an approximation to $y(3)$

SOLUTION: $t_0 = 1, t_1 = 2, t_2 = 3$, so $y(3) = y(t_2) \cong y_2$. The rest is easy.

5. (20 points) The velocity of a skydiver changes at a rate proportional to the difference between their velocity and the so called "terminal velocity" v_T, where $v_T > 0$. If the divers velocity at $t = 0$ is -1, and their velocity at $t = 5$ is -10, find an expression for their velocity at any time as a function of v_T.

SOLUTION: As in previous homework and example problems, the first sentence is interpreted as

$$v' = r(v - v_T)$$

So we have two constants (so far) , one is v_T, which is given and we will not solve for, the other is r, which will be solved for (in terms of v_T). In solving this equation, we get another constant, this constant is solved for using the initial value of v. We solve for r by using the two data points. The final answer is:

$$v = v_T - (1 + v_T)e^{\frac{1}{5}ln(\frac{v_T+10}{v_T+1})t}$$

You may notice that this final answer can be derived without any knowledge of what v_T is. Close inspection of the answer reveals that it is defined only for $-1 < v_T < -10$, and makes physical sense only if $v_T < -10$.