Definition. *We say a function f defined on $[0,\infty)$ is of exponential order if there exist constants $K > 0$, and $a \in \mathbb{R}$ such that $|f(t)| \leq Ke^{at}$.*

Theorem 1. *If f is of exponential order with constants K and a, then the Laplace transform of f is defined for $s > a$.*

Proof. For $s > a$,

$$\left|\int_0^\infty f(t)e^{-st}\,dt\right| \leq \int_0^\infty \left|f(t)e^{-st}\right|\,dt = \int_0^\infty |f(t)|e^{-st}\,dt \leq \int_0^\infty Ke^{at}e^{-st}\,dt$$

$$= K\int_0^\infty e^{(a-s)t}\,dt < \infty$$

The first inequality is clear if you remember that an integral is an area under a curve (draw the curve). This last inequality was shown in class.
The above inequalities show that the absolute value of the Laplace Transform is less than infinity. This means the Laplace transform is defined. □

Now to do #25, you should use the inequalities that I used in the above proof (justify them to yourself), and also the following:

Theorem 2. *Suppose*

$$0 \leq g(s) \leq h(s), \text{ then}$$

$$0 \leq \lim_{s\to\infty} g(s) \leq \lim_{s\to\infty} h(s)$$

In particular,

$$\lim_{s\to\infty} h(s) = 0 \Rightarrow \lim_{s\to\infty} g(s) = 0$$

Theorem 3. *For any function g,*

$$\lim_{s\to\infty} g(s) = 0 \Longleftrightarrow \lim_{s\to\infty} |g(s)| = 0$$

Note: The technique used in the preceding proof is very useful in applied mathematics. It is common that you want to show something is small (some error say), but you cannot figure out exactly what it is. You can often show that it is less than or equal to something else that is small.