Math 307 Final B, Fall 2006

Name: ____

You have 1 hour and 50 minutes to complete the following exam. No notes or calculators allowed. If you are stuck on one problem, I think your best choice is to continue on, attempt other problems, and return to the difficult problem later.

You may make use of the following:

- **Integrating Factors** Given the following equation

  $$y' + p(t)y = g(t),$$

  the appropriate integrating factor is: $\mu(t) = e^{\int p(t)\,dt}$.

- **Euler's Method** Given the following initial value problem

  $$y' = f(t, y)$$
  $$y(t_0) = y_0$$

  Euler's method approximates a solution using:

  $$y(t+h) \cong y(t) + hf(t, y)$$

Theorem 1. *Suppose that f and f' are continuous on $[0, \infty)$. Suppose further that there exist constants K and a such that $|f(t)| \leq Ke^{at}$ for $t \geq 0$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and moreover*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Theorem 2. *Suppose that $f, f', \ldots f^n$ are continuous on $[0, \infty)$. Suppose further that there exist constants K and a such that $|f(t)| \leq Ke^{at}$, $|f'(t)| \leq Ke^{at}, \ldots |f^{(n-1)}(t)| \leq Ke^{at}$ for $t \geq 0$. Then $\mathcal{L}\{f^n(t)\}$ exists for $s > a$, and moreover*

$$\mathcal{L}\{f^n(t)\} = s^n\mathcal{L}\{f(t)\} - s^{(n-1)}f(0) - \ldots sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Theorem 3. *If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a positive constant, then*

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s), \quad s > a$$

Theorem 4. *If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a constant,then*

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \quad s > a + c$$

Theorem 5. *If $F(s) = \mathcal{L}\{f(t)\}$, exists for $s > a$, then*

$$F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}$$

exists and is defined for $s > a$

Theorem 6. *Suppose $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > c$, and $a > 0$. Then*

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right)$$

and the transform $F(as+b)$ exists for $as + b > c$.

1. (10 points) Use an integrating factor to solve the following initial value problem

$$
\begin{aligned}
y' + ty &= te^{-t^2/2} \\
y(0) &= 0
\end{aligned}
$$

Solution: A correct integrating factor is $e^{t^2/2}$, the answer is $y = \frac{t^2}{2}e^{-t^2/2}$.

2. (10 points) Find a solution to the following initial value problem. Your solution should be valid for $t \geq 0$. Use a Laplace transform.

$$y' + y = f(t)$$
$$y(0) = 0,$$
$$where$$
$$f(t) = \begin{cases} 1, & t \in [1, 2) \\ 0, & \text{otherwise} \end{cases}$$

Solution: Writing $f(t) = u_1(t) - u_2(t)$ and taking a Laplace transform, we have

$$Y(s)(1+s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

i.e.

$$Y(s) = (e^{-s} - e^{-2s})\left(\frac{1}{s} - \frac{1}{s+1}\right) = (e^{-s} - e^{-2s})\mathcal{L}\left\{1 - e^{-t}\right\}$$

Using theorem 3 we therefore have

$$y(t) = u_1(t)(1 - e^{-(t-1)}) - u_2(t)(1 - e^{-(t-2)})$$

3. (10 points) Find a solution to the following initial value problem. Your solution should be valid for all $t \in \mathbb{R}$.

$$\begin{aligned} y'' - y &= -2\sin t \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

Solution: $y(t) = -\frac{1}{2}e^{t} + \frac{1}{2}e^{-t} + \sin t$

4. (10 points) Find a solution to the following initial value problem. Your solution should be valid for all $t \in \mathbb{R}$.

$$y' - y = -2e^{-t}$$
$$y(0) = 2$$

Solution: $y(t) = e^t + e^{-t}$

5. (10 points) Use Euler's method with step size= 3 to estimate $y(7)$ when

$$\begin{aligned} y' - \cos(\pi t)y &= 0 \\ y(1) &= 1 \end{aligned}$$

Solution: $t_0 = 1, t_1 = 4, t_2 = 7$, so we will find y_2 using $y_{k+1} = (1 + 3\cos(\pi t_k))y_k$. We get $y_0 = 1, y_1 = -2, y_2 = -8$

6. (10 points) Prove (part of) the following theorem. *You do not need to prove where $F(sa+b)$ exists.*

Theorem. *Suppose $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > c$, and $a > 0, b \neq 0$. Then*

$$F(as+b) = \mathcal{L}\left\{\frac{1}{a}e^{-bt/a} f\left(\frac{t}{a}\right)\right\}$$

and the transform $F(as+b)$ exists for $as+b > c$.

7. (10 points) Using <u>only</u> theorem 5, theorem 3, and the fact that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, find

$$\mathcal{L}\{u_2(t)(t-2)e^{t-2}\}$$

Hint: First find $\mathcal{L}\{te^t\}$.

Solution: Using theorem 5, $\mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}$. Now put $f(t) = te^t$, then $\mathcal{L}\{u_2(t)(t-2)e^{t-2}\} = u_2(t)f(t-2)$. Therefore our answer is $\frac{e^{-2s}}{(s-1)^2}$.