Math 427 Homework #1 Solutions

Thomas Sixuan Lou

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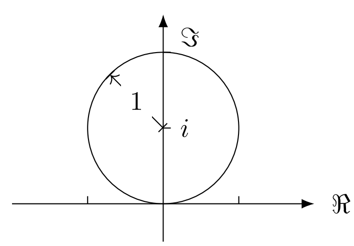
Problem 1.1.5.

Proof. Suppose $z = a + ib$ satisfies $z^2 = i$, then expanding the equation and equating the real and imaginary parts we get relations $a^2 = b^2$ and $2ab = 1$. The first relation is equivalent to $a = \pm b$, substituting it into the second one we get $a^2 = \pm 1/2$ so $a = \pm\sqrt{2}/2$. Finally, we check that the only possibilities of $(a, b) = (\pm\sqrt{2}/2, \pm\sqrt{2}/2)$ that give actual solutions are when a and b have the same solution, that is $(a, b) = (\sqrt{2}/2, \sqrt{2}/2)$ or $(a, b) = (-\sqrt{2}/2, -\sqrt{2}/2)$. Therefore, the solutions are $z = \sqrt{2}/2 + \sqrt{2}/2i$ and $z = \sqrt{2}/2 + \sqrt{2}/2i$. □

Problem 1.1.13.

Proof.



□

Problem 1.1.17.

Geometric Proof. The set of points in $\mathbb{C}$ with modulus 1 are precisely the set of points in the complex plane that is of distance 1 to the origin. Therefore any such point may be parametrized by $\cos\theta + i\sin\theta$ for some $\theta \in \mathbb{R}$. □

Algebraic Proof. Suppose $z = a + ib \in \mathbb{C}$ has $|z| = 1$, then $a^2 + b^2 = 1$. Since $0 \leq a^2, b^2$, we must have $0 \leq a^2, b^2 \leq 1 \implies 0 \leq |a| \leq 1$. Therefore we may pick a $0 \leq \theta < 2\pi$ with $\cos\theta := a$ and such that the $\sin\theta$ as the same sign as b. By substituting $a = \cos\theta$, we get the equation $\cos^2\theta + b^2 = 1$ which implies that $b^2 = \sin^2\theta$. Thus, $b = \pm\sin\theta$ be we've already arranged that b has the same sign as $\sin\theta$. We conclude that $b = \sin\theta$. This proves the claim. □

Problem 1.2.4.

Geometric Proof. Note $1/\sqrt{2} + i/\sqrt{2} = \cos\pi/4 + i\sin\pi/4 = \exp(i\pi/4)$, hence the n-th power is $\exp(i\pi n/4)$. In another word, each such point lies on the unit circle and increasing the exponent by 1 amounts to rotate it counterclockwise $\pi/4$ radians. Therefore the sequence does not converge to anything.

Algebraic Proof. Observe powers of $1/\sqrt{2} + i/\sqrt{2}$ is periodic in the finite sequence

$$\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}},\ i,\ -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}},\ -1,\ -\frac{1}{\sqrt{2}} + -\frac{i}{\sqrt{2}},\ -i,\ \frac{1}{\sqrt{2}} + -\frac{i}{\sqrt{2}},\ 1$$

Hence no limit exist.

Problem 1.5. Prove that the sequence $(6 - ni)^{-1}$ converges to 0.

Proof. First observe for $z \in \mathbb{C}$, $|1/z| = \left|\overline{z}/|z|^2\right| = |\overline{z}|/|z|^2 = 1/|z|$. Therefore

$$\lim_{n\to\infty}\left|\frac{1}{6-in}\right| = \lim_{n\to\infty}\frac{1}{\sqrt{36+n^2}} \leq \lim_{n\to\infty}\frac{1}{n} = 0$$

where we have used that $\frac{1}{36+n^2} \leq \frac{1}{n}$. □

Problem 1.2.12.

Proof. For each term of the series to be well-defined $z^2 \neq -n^2 \implies z \neq in$ for any $n \in \mathbb{N}$. Suppose $z \in \mathbb{C}$ is a complex number which is not equal to in for any $n \in \mathbb{N}$, that is each term in the series is well-defined. We claim the series converges absolutely by using the limit comparison test: if $\{a_n\}$ and $\{b_n\}$ are sequences of positive real numbers such that $\lim_{n\to\infty}\frac{a_n}{b_n} => 0$, then $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} b_n$ converges.

Let us compare the series $\sum_{n=0}^{\infty} 1/\left|n^2+z^2\right|$ with $\sum_{n=0}^{\infty} 1/n^2$, the latter of which we know is convergent. Suppose $z = a + ib$ then the limit of the ratios of the terms in the series is

$$\lim_{n\to\infty}\frac{n^2}{|n^2+z^2|} = \lim_{n\to\infty}\frac{n^2}{\sqrt{n^4+2n^2(a^2-b^2)+a^4+b^4+2a^2b^2}}$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{1+2(a^2-b^2)/n^2+(a^4+b^4+2a^2b^2)/n^4}} = 1$$

Since $\sum_{n=0}^{\infty} 1/n^2$ coverges, so does $\sum_{n=0}^{\infty} 1/\left|n^2+z^2\right|$ for all $z \in \mathbb{C}$ where $z \neq in$ for any $n \in \mathbb{N}$. □

Problem 1.3.3.

Proof.

$$\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = e^{i\pi/4}$$

□

Problem 1.10. Let $\{z_n\}$ and $\{w_n\}$ be sequences such that $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$. Show that $\lim_{n\to\infty}(z_n w_n) = zw$.

Proof.
We will first prove the following claim:

Claim: Given squences $\{a_n\}$ and $\{b_n\}$ of real numbers with $\lim_{n\to\infty} a_n \to 0$ and $\lim_{n\to\infty} b_n \to 0$, then $\lim_{n\to\infty} a_n b_n = 0$.
Proof of Claim: Let $\epsilon > 0$ be given, there exists N_1 and N_2 such that $|a_n| < \epsilon$ and $|b_n| < 1$ for all $n > \max\{N_1, N_2\}$. Therefore, we have $|a_n b_n| = |a_n|\,|b_n| < \epsilon$ for $n \geq \max\{N_1, N_2\}$. This proves the claim.

Let $\{z_n\}$ and $\{w_n\}$ be sequences such that $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$. We know $\lim_{n\to\infty}|z_n - z| = 0$ and $\lim_{n\to\infty}|w_n - w| = 0$. By the claim above, we have

$$\lim_{n\to 0}|z_n - z|\,|w_n - w| = 0.$$

By triangle inequality, for each n, we have the inequality

$$\begin{aligned}|(z_n - z)(w_n - w)| &= |(z_n w_n - zw) - (z_n w - zw) - (zw_n - zw)| \\ &\geq |z_n w_n - zw| - |z_n w - zw| - |zw_n - zw|\end{aligned}$$

This implies that for all n, we have

$$|z_n w_n - zw| \leq |z_n - z|\,|w_n - w| + |z_n w - zw| + |z w_n - zw|$$

Since each summand on the right has limit 0, that is

$$\lim_{n\to\infty} |z_n - z|\,|w_n - w| = \lim_{n\to\infty} |z_n w - zw| = \lim_{n\to\infty} |z w_n - zw| = 0,$$

we may conclude that $\lim_{n\to\infty} |z_n w_n - zw| = 0$, as desired. □