Math 427 Homework #4 Solutions

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Problem 4.1. Taylor 2.3.2.

Solution: Recall we defined the complex function sin by

$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2i}.$$

Therefore, by definition of the Riemann integral of complex-valued functions, we can rewrite the integral of interest as

$$\int_0^1 \sin(it)dt = \frac{1}{2i}\int_0^1 e^{-t} - e^t dt,$$

where the integral $\int_0^1 e^{-t} - e^t dt$ is an ordinary Riemann integral of real-valued function. Therefore

$$\int_0^1 \sin(it)dt = -\frac{i}{2}\Big[-e^{-t} - e^t\Big]_0^1 = -\frac{i}{2}\Big(-\frac{1}{e} - e + 2\Big) = i(\frac{1}{2e} + \frac{e}{2} - 1).$$

□

Problem 4.2. Taylor 2.3.5.

Solution:

(a) Recall if the path γ traces over a circle in $\mathbb{C}$ centered at the origin of radius r, counterclockwise once from time 0 to 2π, then it may be parametrized by the function $\gamma(t) = re^{it}$ for $0 \le t \le 2\pi$. If we translate the circle to be centered at $z_0 \in \mathbb{C}$ instead, then the parametrization is also translated to $\gamma(t) = z_0 + re^{it}$, for $0 \le t \le 2\pi$.

(b) Based on part (a), if we also want to reverse the direction of the path, we will want the parameter to go from 2π back to 0. Therefore we may define the new parametrization by change of variable $s = 2\pi - t$, yielding $\gamma(s) = z_0 + re^{is}$, where $s = 2\pi - t$, and t ranges from 0 to 2π. Since $e^{i(2\pi - t)} = e^{-it}$, this is equivalent to the parametrization $\gamma(t) := z_0 + re^{-it}$ where t ranges from 0 to 2π.

(c) Based on part (a), if we also want to speed up how the parametrization traces over the circle to trace 3 times around the circle $\{|z - z_0| = r : z \in \mathbb{C}\}$ from time 0 to 2π. We will want the parameter to go 3 times faster, hence we may define the new parametrization by change of variable $s = 3t$, yielding $\gamma(s) := z_0 + re^{is}$, where $s = 3t$, and t ranges from 0 to 2π. This is equivalent to the parametrization $\gamma(t) := z_0 + re^{i3t}$ where t ranges from 0 to 2π.

□

Problem 4.3. Taylor 2.3.7.

Proof. Recall the fundamental theorem of calculus for real-valued functions: if f is a real valued function continuous on the interval $[a, b]$, and it has F as its primitive on (a, b) (this means F is differentiable on (a, b), with $F'(t) = f(t)$ for all $t \in (a, b)$), then

$$\int_a^b f(t)dt = F(b) - F(a).$$

Now let f be a smooth complex-valued function on an interval $[a, b]$, we can write it as $f(t) = u(t)+iv(t)$, where u and v are real-valued functions that are smooth on (a, b). Since u and v are smooth, it has continuous derivatives of all orders, in particular $u'(t)$ and $v'(t)$ are both continuous on (a, b). And we may define their value at the end points of $[a, b]$ by taking the one sided limits so they are continuous on the whole interval. Then by the fundamental theorem of calculus, and the definition of integrating complex-valued functions on an interval,

$$\begin{aligned}\int_a^b f'(t)dt &= \int_a^b u'(t) + iv'(t)dt = \int_a^b u'(t)dt + i\int_a^b v'(t)dt \\ &= u(b) - u(a) + i(v(b) - v(a)) = (u(b) + iv(b)) - (u(a) + iv(a)) = f(b) - f(a).\end{aligned}$$

□

Problem 4.4. Taylor 2.3.13.

Proof. No, we will present a counterexample. Consider the function $f(z) = 1/z$. Since z is analytic and nonzero on $\mathbb{C} \setminus \{0\}$, the quotient is analytic on $\mathbb{C} \setminus \{0\}$. We will compute the integral of it when we trace over the unit circle in $\mathbb{C}$. Consider the parametrization given by $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$. On the one hand,

$$Re\Big[\int_{|z|=1} \frac{1}{z}dz\Big] = Re\Big[\int_0^{2\pi} e^{-it}(ie^{it})dt\Big] = Re\Big[\int_0^{2\pi} idt\Big] = Re[2\pi i] = 0.$$

But on the other hand,

$$\int_{|z|=1} Re\Big[\frac{1}{z}\Big]dz = \int_0^{2\pi} Re\Big[e^{-it}\Big](ie^{it})dt = -\int_0^{2\pi} \cos(t)\sin(t)dt + i\int_0^{2\pi} \cos^2(t)dt = i\pi.$$

□

Problem 4.5. Taylor 2.4.2.

Solution: Let γ trace around the unit circle twice in the counterclockwise direction, from time 0 to 2π. Consider the parametrization given by $\gamma(t) = e^{-2it}$ for $0 \le t \le 2\pi$. Then by the definition of contour integral,

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} e^{2it}(2ie^{-2it})dt = \int_0^{2\pi} 2idt = 4\pi i.$$

□

Problem 4.6. Taylor 2.4.10.

Proof. Write the polynomial as $p(z) = a_n z^n + \cdots + a_1 z + a_0$. If we show that $\int_\gamma z^m = 0$ for each integer $m \ge 0$, then by Theorem 2.4.6 in Taylor, we may conclude that

$$\int_\gamma p(z) = a_n \int_\gamma z^n + \cdots + a_0 \int_\gamma z^0 = a_0 \cdot 0 + \cdots + a_0 \cdot 0 = 0.$$

Therefore it is suffices to show $\int_\gamma z^m = 0$ for all integer $m \ge 0$. Let integer $m \ge 0$ be fixed,

$$\int_\gamma z^m = \int_0^{2\pi} e^{imt}(ie^{it})dt = i\int_0^{2\pi} e^{i(m+1)t}dt = i\Big[\frac{e^{i(m+1)t}}{(m+1)i}\Big]_0^{2\pi} = \frac{1}{m+1}\Big(e^{i(m+1)2\pi} - 1\Big) = 0.$$

This proves the statement. □

Problem 4.7. Taylor 2.6.3.

Solution: Observe since the function e^z is analytic on $\mathbb{C}$, the function $1 - e^z$ is also analytic on $\mathbb{C}$. It is zero only when $e^z = 1$, which happens if and only if $z = 2\pi ki$ for some $k \in \mathbb{Z}$. Therefore we conclude the function $(1 - e^z)^{-1}$ is analytic on $\mathbb{C} \setminus \{2\pi ki : k \in \mathbb{Z}\}$. In particular, it is analytic on the strip $\{x + iy : y \in (0, 2\pi), x \in \mathbb{R}\}$, which is a convex open set. Since the path γ is contained entirely in the strip (because γ is a circle centered at $2i$ with radius 1). By Cauchy's Integral Theorem, it follows

$$\int_\gamma \frac{dz}{1 - e^z} = 0.$$

□

Problem 4.8. Taylor 2.6.5.

Solution: Let γ be a closed curve in $\mathbb{C} \setminus \{0\}$, by Jordan Curve Theorem, the (image of the) curve divides the complex plane into two connected components, one is bounded, the other is unbounded, let D denote the bounded component together with the boundary (thus $\gamma = \partial D$). We consider two cases: (1) $0 \notin D$ and (2) $0 \in D$.

1. Suppose $0 \notin D$, define subsets of D (illustrated in Figure 1):

$$\begin{aligned} D_1 &:= D \cap \{x + iy : x + iy \neq 0, x \geq 0, y \geq 0\} \\ D_2 &:= D \cap \{x + iy : x + iy \neq 0, x \leq 0, y \geq 0\} \\ D_3 &:= D \cap \{x + iy : x + iy \neq 0, x \leq 0, y \leq 0\} \\ D_4 &:= D \cap \{x + iy : x + iy \neq 0, x \geq 0, y \leq 0\}. \end{aligned}$$

In other words, D_i, is the part of D that is in the i-th quadrant. Since if two quadrants are adjacent, the two D_i's share a same boundary edge. Therefore, the contour integral of $1/z^2$ over ∂D is exactly the sum of contour integrals of $1/z^2$ over each ∂D_i.

Observe each D_i is contained in a convex open set on which $1/z^2$ is analytic. Explicitly, D_1 is contained in the convex open set $\{x + iy : y > -x\}$; D_3 is contained in $\{x + iy : y < -x\}$; D_2 is contained in $\{x + iy : y > x\}$ and D_4 is contained $\{x + iy : y < x\}$. Therefore by Cauchy's Integral Theorem, $\int_{\partial D_i} 1/z^2 dz = 0$ for all i, hence

$$\int_{\partial D} \frac{dz}{z^2} = \sum_{i=1}^{4} \int_{\partial D_i} \frac{dz}{z^2} = 0.$$

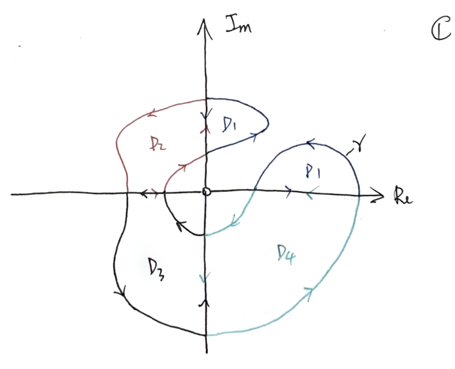

Figure 1: Case 1

2. Suppose $0 \in D$, since 0 is not in the image of γ, we may find a closed ball $B_\epsilon(0) = \{z \in \mathbb{C} : |z| \le \epsilon\}$ that is contained entirely in D. Define regions:

$$\begin{aligned} D_1 &:= D \cap \{x+iy : |x+iy| \ge \epsilon, x \ge 0, y \ge 0\} \\ D_2 &:= D \cap \{x+iy : |x+iy| \ge \epsilon, x \le 0, y \ge 0\} \\ D_3 &:= D \cap \{x+iy : |x+iy| \ge \epsilon, x \le 0, y \le 0\} \\ D_4 &:= D \cap \{x+iy : |x+iy| \ge \epsilon, x \ge 0, y \le 0\} . \end{aligned}$$

Since if two quadrants are adjacent, the corresponding D_i's share a same boundary edge. Therefore,

$$\int_{\partial D} \frac{dz}{z^2} = \sum_{i=1}^{4} \int_{\partial D_i} \frac{dz}{z^2} + \int_{|z|=\epsilon} \frac{dz}{z^2},$$

where we travel the contour $|z| = \epsilon$ counterclockwise, we may parametrize it with $\Gamma(t) := \epsilon e^{it}$ for $0 \le t \le 2\pi$. By the same argument as case 1, we conclude $\int_{\partial D_i} 1/z^2 dz = 0$ for all i, therefore

$$\int_{\partial D} \frac{dz}{z^2} = \int_{|z|=\epsilon} \frac{dz}{z^2} = \int_0^{2\pi} \frac{\epsilon i e^{it}}{\epsilon^2 e^{i2t}} dt = \frac{i}{\epsilon} \int_0^{2\pi} e^{-it} dt = 0.$$

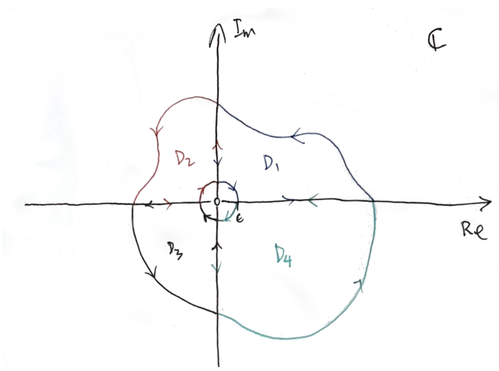

Figure 2: Case 2

In either case we discover that the contour integral of $1/z^2$ over γ is 0.

□

Problem 4.9.

1. Provide a parameterization $\gamma(t)$ of a path in $\mathbb{C}$ which traces clockwise the triangle with vertices $(0, -1)$, $(1, 1)$ and $(-1, 1)$.

2. Compute explicitly the integral
$$\int_\gamma \frac{dz}{z}.$$

3. How does your answer agree with the conclusion of Cauchy's Integral Theorem?

Proof.

1. Since contour integral is path additive (Theorem 2.4.6 of Taylor), we divide the contour into three parts, the first one travels from $-1+i$ to $1+i$, the second one travels from $1+i$ to $-i$, and the third

travels from $-i$ back to $-1+i$. We denote their parametrizations γ_1, γ_2 and γ_2 respectively. More explicitly, they are defined as follows:

$$\gamma_1(t) = t + (-1+i), \quad 0 \le t \le 2$$
$$\gamma_2(t) = (1+i) + t(-1-2i), \quad 0 \le t \le 1$$
$$\gamma_3(t) = -i + t(-1+2i), \quad 0 \le t \le 1.$$

2. Recall $1/z$ is analytic on $\mathbb{C} \setminus \{0\}$, and it has $\log_I(z)$ as a primitive, where the choices of branch I depend on the domain of interest. For example, on the set $\mathbb{C} \setminus (-\infty, 0]$, the function $1/z$ has the principal branch of $\log(z)$ as a primitive. Therefore we can apply Theorem 2.5.6 to conclude

$$\int_{\gamma_1} \frac{dz}{z} = \log(1+i) - \log(-1+i) = \log(\sqrt{2}) + i\frac{\pi}{4} - \log(\sqrt{2}) - i\frac{3\pi}{4} = -\frac{\pi i}{2},$$

and

$$\int_{\gamma_2} \frac{dz}{z} = \log(-i) - \log(1+i) = \log(1) + i\Big(-\frac{\pi}{2}\Big) - \log(\sqrt{2}) - i\frac{\pi}{4} = -\log(\sqrt{2}) - \frac{3\pi i}{4}.$$

For the integral over γ_3, since the path γ_3 crosses the negative real line, we may not use the principal branch of log. Instead, we use $\log_I(z)$, where $I = (0, 2\pi]$ (the cut line of this branch of log is the line $[0, \infty)$). Therefore

$$\int_{\gamma_3} \frac{dz}{z} = \log_I(-1+i) - \log_I(-i) = \log(\sqrt{2}) + i\frac{3\pi}{4} - \log(1) - i\frac{3\pi}{2} = \log(\sqrt{2}) - \frac{3\pi i}{4}.$$

Therefore, we conclude

$$\int_{\gamma} \frac{dz}{z} = \sum_{i=1}^{3} \int_{\gamma_3} \frac{dz}{z} = -\frac{\pi i}{2} - \log(\sqrt{2}) - \frac{3\pi i}{4} + \log(\sqrt{2}) - \frac{3\pi i}{4} = -2\pi i.$$

3. Cauchy's Integral Theorem does not apply here since (1) the open set $\mathbb{C} \setminus \{0\}$ is not a convex open set and (2) there's no way to define the value of $1/z$ at the origin making the function continuous at 0 (too see this, notice $1/z$ is unbounded as z approaches the origin).

□

Problem 4.10. Taylor 2.6.12.

Solution: Recall the exponential function e^z is analytic on the entire complex plane (which is convex). Therefore by Cauchy's Integral Formula,

$$\mathrm{Ind}_{|z|=1}(0) \cdot e^0 = \frac{1}{2\pi i} \int_{|z|=1} \frac{e^z}{z-0} dz.$$

Therefore the integral of interest is:

$$\int_{|z|=1} \frac{e^z}{z} dz = 2\pi i \, \mathrm{Ind}_{|z|=1}(0).$$

To compute the index $\mathrm{Ind}_{|z|=1}(0)$ we choose the parametrization $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$,

$$\mathrm{Ind}_{|z|=1}(0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dw}{w} = \frac{1}{2\pi i} \int_0^{2\pi} i dz = 1.$$

Therefore,

$$\int_{|z|=1} \frac{e^z}{z} dz = 2\pi i.$$

□