Introduction to stacks & moduli

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Course goal: Introduce alg. stacks with moduli in mind.

THM: The moduli space $\overline{M}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper and irreducible Deligne-Mumford stack of dim $= 3g-3$ which admits a proj. coarse mod. space.

Course details:

- around 20 lectures, each 80 min
- lectures will be recorded
- Questions welcome (save longer questions until end)
- discord/zulip ?

References

- course lecture notes (on website)

Stacks

- Champs algébriques, Laumon & Moret-Bailly
- Algebraic spaces, Knutson
- Algebraic spaces and stacks, Olsson
- Notes on Grothendieck topologies, fibered categories and descent theory, Vistoli
- Stacks project

Curves

- The irreducibility of the space of curves of given genus – Deligne and Mumford
- Moduli of curves, Harris and Morrison
- Projectivity of complete moduli, Kollár
- Stacks project

Advice:

- Learning the theory of stacks and moduli is hard! Requires work!
- Requires simultaneously absorbing
  - functorial approach in AG
  - working with groupoids & stacks
  - replacing Zariski topology with étale topology
  - systematic use of descent theory
  - several advanced topics not usually covered in a first course in AG such as
    - properties of flat, étale, smooth maps
    - existence of Hilbert/Quot schemes
    - algebraic groups & actions
    - deformation theory
    - Artin approximation
    - birational geometry of surfaces

Tips

* Work through material yourself
- work out your own examples
- do exercises
- complete details in proofs

* Don't read the notes linearly

* Accept that there are topics you don't know & still won't in 10 weeks. But try to use such results anyway!

* Try to find a balance

accepting & using advanced results $\longleftrightarrow$ understanding why such results hold

accepting details $\longleftrightarrow$ checking details

Have faith that either
- ↳ you could work out details
- ↳ at some point later, you will properly learn the material

Today: motivation & history

§1. Moduli

Let $*$ be your favorite mathematical object

"DEF": A moduli space of $*$ is a space M such that

$$\text{pts of } M \xleftrightarrow{\text{bij}} \text{isom. classes of } *$$

Ex 1 $*$ = smooth, conn, proj curves of genus g

$\rightsquigarrow M_g$ moduli of smooth curves

Ex 2 $*$ = plane curves

$$M = \{ C \subset \mathbb{P}^2 \text{ deg } d \}/\sim \;=\; \mathbb{P}\Gamma(\mathbb{P}^2, \mathcal{O}(d))/PGL_2$$

where $C \sim C'$ if proj. equiv.

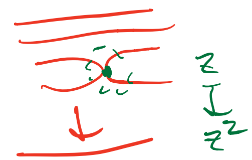
i.e. $\exists$
$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\sim} & \mathbb{P}^2 \\ \cup & & \cup \\ C & \xrightarrow{\sim} & C' \end{array}$$

Other choices for $\sim$

- abstract isom
- $C \sim C'$ if C & C' are equal as subschs of $\mathbb{P}^2$

Ex 3 Hurwitz moduli space

Instead of studying C in $\mathbb{P}^n$, study branched covers $C \to \mathbb{P}^1$ deg d



$$Hur_{d,g} := \left\{ C \xrightarrow{\text{deg } d} \mathbb{P}^1 \;\middle|\; C \text{ sm, conn, proj of genus } g \right\}/\sim$$

$$[C \to \mathbb{P}^1] \sim [C' \to \mathbb{P}^1] \iff \begin{array}{ccc} C & \xrightarrow{\sim} & C' \\ & \searrow \; \swarrow & \\ & \mathbb{P}^1 & \end{array}$$

Ex 4 Vector bundles on a curve

Fix C sm, conn, proj curve

$$M_{C,r,d} = \left\{ \begin{array}{l} \text{vect. bdls on } C \\ \text{of rank } r, \text{ deg } d \end{array} \right\}/\sim$$

⚠ So far, only defined as sets

$\boxed{\text{MODULI MIRACLE}}$

These spaces are alg. varieties!

§2 History

Riemann (1857) The "number of moduli" of smooth curves of genus g is $3g-3$

- Gave several arguments. Here's one

$d \gg 0$

$$\begin{tikzcd}
\mathrm{Hur}_{d,C} \arrow[r, hook] \arrow[d] & \mathrm{Hur}_{d,g} \arrow[r, "\text{dense image}", "\text{finite fibers}"'] \arrow[d] & \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1 \\
{[C]} \arrow[r, phantom, "\in"] & M_g &
\end{tikzcd}$$

moduli $C \to \mathbb{P}^1$

$[C \to \mathbb{P}^1] \mapsto$ branched points

$[C \to \mathbb{P}^1] \mapsto [C]$

Riemann-Hurwitz: $2g(C)-2 = d(2g(\mathbb{P}^1)-2)+R$

$\Rightarrow$ # branched pts $= R = 2d+2g-2$

Thus $\dim \mathrm{Hur}_{d,g} = 2d+2g-2$

$$= \underbrace{\dim M_g}_{\text{Want}} + \underbrace{\dim \mathrm{Hur}_{d,C}}_{?}$$

Fix C. A degree d map $C \to \mathbb{P}^1$ is defined by line bdl L of deg $= d$ & 2 sections mod scalars

$L \in \mathrm{Pic}_d(C) \leftarrow \mathrm{Sym}^d C$ $\qquad 2h^0(L)-1 \overset{RR}{=} \boxed{2(d+1-g)-1}$

$\boxed{\dim g}$ $\qquad$ line bdl + section/scaling

$h^0(L)-1 \overset{RR}{=} d-g$

Conclude: $\dim \mathrm{Hur}_{d,C} = 2d-g+1$

$\Rightarrow \dim M_g = 3g-3$

- Riemann called M_g "mannigfaltigkeiten" (manifold-like)

  manifolds weren't introduced until 1940s

- Weil 1958: "As for M_g, there is virtually no doubt that it can be provided the structure of an alg. variety."
- Grothendieck 1960: Aware M_g not representable but showed representability of the moduli of smooth curves with level n structure (i.e. C + basis of $H_1(C, \mathbb{Z}/n)$)

  struggled with projectivity

- Mumford 1965: M_g is a variety

Mumford used GIT

$\boxed{\text{Geometric Invariant Theory}}$

There are other approaches including analytic or topological

Our approach will be entirely algebraic integrating ideas in GIT with stack theory

1. Show $\overline{\mathcal{M}}_g$ is proper Deligne-Mumford stack
2. Use Keel-Mori theorem to show $\exists$ coarse mod space $\overline{\mathcal{M}}_g \longrightarrow \overline{M}_g$ proper alg. space
3. Show a line bdl on $\overline{\mathcal{M}}_g$ descends to an ample line bdl

Kollár

§3. Functorial Worldview

Grothendieck:

$\boxed{\text{Spaces are functors}}$

Let $\boxed{\text{Sch = category of schemes}}$

We are interested in contravariant functors

$\boxed{F : Sch \to Sets}$

Ex 1: If X is a scheme,

$h_X : Sch \to Sets, \quad S \mapsto Mor(S, X)$

$\boxed{\text{Yoneda's lemma}}$ ("h_X determines X")

For any contravar. functor $G : Sch \to Sets$

$$Mor(h_X, G) \xrightarrow{\sim} G(X) \quad \text{bijective}$$

$$(\alpha : h_X \to G) \longmapsto \alpha_X(id_X)$$

nat trans

Idea: $X \overset{set}{=} \coprod_{k \text{ field}} h_X(\operatorname{Spec} k) / \sim$

Ex 2 Projective space $\mathbb{P}^n$ represents

$F : Sch \to Sets$,

$$S \mapsto \left\{ \begin{array}{l} \bullet \text{ line bdl } L \text{ on } S \\ \bullet \text{ sections } s_0, \dots, s_n \in \Gamma(S, L) \end{array} \;\middle|\; \begin{array}{l} \text{the } s_i \\ \text{generate } L \end{array} \right\}$$

i.e. $Mor(S, \mathbb{P}^n) \xrightarrow{\sim} F(S)$

$$Mor(S,\mathbb{P}^n) = h_{\mathbb{P}^n}(S)$$

means $\mathcal{O}_S^{n+1} \xrightarrow{s_0, \dots, s_n} L$ surjective

Ex 3 Grassmanian functor

$$Gr(k,n) : Sch \to Sets$$

$$S \mapsto \{ \mathcal{O}_S^{\oplus n} \twoheadrightarrow V \mid V \text{ vect. bdl of rank } k \}$$

This functor is represented by a scheme — which we also denote as $Gr(k,n)$ — which is projective over $\mathbb{Z}$

⚠ Conflate X with h_X

Ex 4 Can also consider functors

$$\boxed{AffSch \xrightarrow{F} Sets}$$

affine schemes

Exercise

A scheme X can be equiv defined as a functor $F : AffSch \to Sets$ where $\exists$ open subfunctors $F_i \subset F$ such that

(a) each F_i is represented by an affine scheme

(b) $\coprod_i F_i \to F$ is surjective.

$h_S \to F$

- $F_i \hookrightarrow F$ open $\iff$ $\forall$ maps $S \to F$ (affine scheme)

$$\begin{array}{ccc} F_i \times_F S & \hookrightarrow & S \\ \downarrow & & \downarrow \\ F_i & \hookrightarrow & F \end{array}$$

open imm. of schemes

fiber product in cat. of functors

set fiber product

$$(F_i \times_F S)(T) = F_i(T) \times_{F(T)} S(T)$$

- $\coprod_i F_i \twoheadrightarrow F \iff \forall$ maps $S \to F$

$\coprod_i F_i \times_F S \twoheadrightarrow S$ as map of schemes

Algebro-geometric space	Type of object	Obtained by gluing
Schemes	sheaf	affine schemes in the Zariski topology
Algebraic spaces	sheaf	affine schemes in the étale topology
Deligne–Mumford stacks	stack	affine schemes in the étale topology
Algebraic stacks	stack	affine schemes in the smooth topology

~~sheaf~~ stack = contr. functor $F : Sch \to \cancel{Sets}$ Groupoids

\+ sheaf axiom $\{U_i\}$ of U

$$0 \to F(U) \to \prod F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

exact

Ex 5 Define functor

$$F_{M_g} : Sch \longrightarrow Sets$$

$$S \longmapsto \left\{ \begin{array}{c} \mathcal{C} \\ \downarrow \text{smooth} \\ S \end{array} \;\middle|\; \begin{array}{l} \forall s : \operatorname{Spec} k \to S \quad k = \bar{k} \\ \mathcal{C}_s \text{ is sm, conn, proj} \\ \text{curve of genus } g \end{array} \right\}$$

$\boxed{\text{FACT} \quad F_{M_g} \text{ is not representable}}$

Principle Let $F : Sch/_{\mathbb{C}} \to Sets$ functor

Let $\mathcal{E} \in F(S)$ be object over $\mathbb{C}$-variety S

For $s \in S(\mathbb{C})$, let $\mathcal{E}_s \in F(\mathbb{C})$ be restriction

$\operatorname{Spec} \mathbb{C} \xrightarrow{s} S$

If (a) all $\mathcal{E}_s$ are isomorphic

(b) $\mathcal{E}$ is not trivial

$\Longrightarrow$ F is not representable by a scheme

Reason: If X (scheme) represents F, then

$\mathcal{E}_s \in F(\mathbb{C}) \longleftrightarrow$ point $x \in X(\mathbb{C})$

Two maps $S \underset{\text{const } x}{\overset{f \;\; \mathcal{E}}{\rightrightarrows}} X$ must be same!

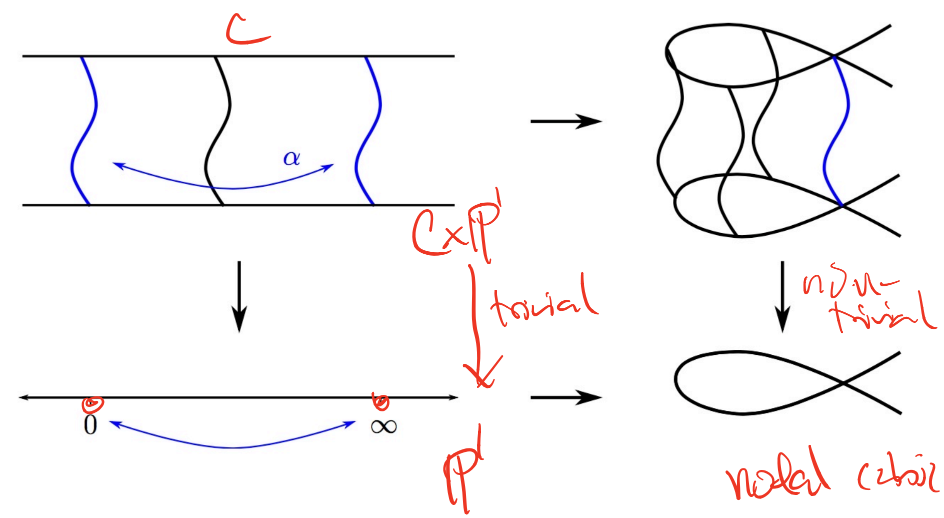
$\Longrightarrow \mathcal{E}$ trivial

$\forall s \in S(\mathbb{C}) \quad f(s) = x$

$\operatorname{Spec} \mathbb{C} \xrightarrow{x} X$

$\boxed{\begin{array}{l} \text{Automorphisms} \\ \rightsquigarrow \text{non-trivial families where} \\ \quad \text{all fibers are isomorphic} \end{array}}$

Example: Let $\boxed{\alpha : C \xrightarrow{\sim} C}$ non-trivial aut



Trichotomy of Moduli

Aut	No Auts	Finite Auts	Infinite Auts
Type of space	Algebraic variety / space	Deligne–Mumford stack	algebraic stack
Defining property	Zariski/étale-locally an affine scheme	étale-locally an affine scheme	smooth-locally an affine scheme
Examples	$\mathbb{P}^n$, $\mathrm{Gr}(k,n)$, Hilb, Quot	$\mathcal{M}_g$	$\mathcal{M}_{C,r,d}$
Quotient stacks $[X/G]$	action is free	finite stabilizers	any action
Existence of moduli varieties / spaces	already an algebraic variety/space	coarse moduli space	good moduli space