III Geometry of DM stacks

TODAY

- Quasi-coherent sheaves
- Local structure

I. Sites, sheaves, & stacks

II Alg spaces & stacks

III Geometry of DM stacks (~2 lectures)

IV Stable curves (~6-7 lectures)

§0. Recap

Theorem (Existence of Miniversal Presentations)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ a finite type point with smooth stabilizer G_x.*
$\Longrightarrow$ *$\exists$ smooth morphism $(U,u) \to (\mathcal{X},x)$ from a scheme of relative dimension $\dim G_x$ s.t.*

$$\begin{tikzcd} \operatorname{Spec}\kappa(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "f"] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X}. \end{tikzcd}$$

In particular, if G_x is finite and reduced, there is an étale morphism $(U,u) \to (\mathcal{X},x)$ from a scheme.

- We showed that
  $T_{U,u} \xrightarrow{\sim} T_{\mathcal{X},f(u)}$ isom.
- If $\mathcal{X}$ is f.type/k & $\mathcal{X}$ is smooth at x,
  $\dim_x \mathcal{X} = \dim T_{\mathcal{X},x} - \dim G_x$

Corollary (Equiv. characterizations of DM stacks)**.** *Let $\mathcal{X}$ be a noetherian alg. stack. The following are equiv:*

(1) $\mathcal{X}$ is Deligne–Mumford;

(2) every point of $\mathcal{X}$ has a finite, reduced stabilizer;

(3) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified.

$\Rightarrow \mathcal{M}_g$ is DM

Smoothness

Theorem (Formal Lifting Criteria). *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then f is smooth if and only if f is locally of finite presentation and for every diagram*

$$\begin{tikzcd}
\operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists"] & \mathcal{Y},
\end{tikzcd}$$

of solid arrows where $A \twoheadrightarrow A_0$ is a surjection of rings with nilpotent kernel, there exists a lifting.

If $\mathcal{X}$ and $\mathcal{Y}$ are noetherian, then it suffices to consider diagrams where A and A_0 are local artinian rings.

Often: There is an "obstruction" to the existence of $\operatorname{Spec} A \to \mathcal{X}$ which is an element of a coh. group

For $\mathcal{M}_g$,

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[dr, hook] \arrow[rr, bend left, "{[C]}"] & & \\
& \operatorname{Spec} A_0 \arrow[r] \arrow[d, "\text{small ext}", "\ker(A \twoheadrightarrow A_0) = k"'] & \mathcal{M}_g \arrow[d] \\
& \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists"] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

local artinian

$\underline{\underline{ob}} \in H^2(C, T_C) = 0$

$\implies \mathcal{M}_g$ smooth $/ \mathbb{Z}$

Separatedness & Properness

We say $\mathcal{X} \to \mathcal{Y}$ is

(1) separated if $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is proper

(2) proper if $\mathcal{X} \to \mathcal{Y}$ is f.type, univ. closed & separated.

Ex: X scheme $\Rightarrow$ Δ_X loc. closed imm

X separated $\Leftrightarrow$ Δ_X closed imm
$\Leftrightarrow$ Δ_X finite
$\Leftrightarrow$ Δ_X proper

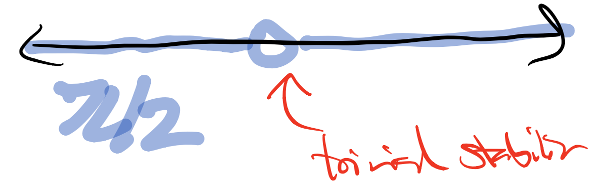
Ex: BG, G finite

BG separated b/c

$$\begin{tikzcd} G \arrow[r, "\text{finite}"] \arrow[d, "\text{ét}"] & \mathrm{Spec}\, k \arrow[d, "\text{ét}"] \\ BG \arrow[r] & BG \times BG \end{tikzcd}$$

Ex: $\mathbb{Z}/2 \curvearrowright \mathcal{U} =$ 

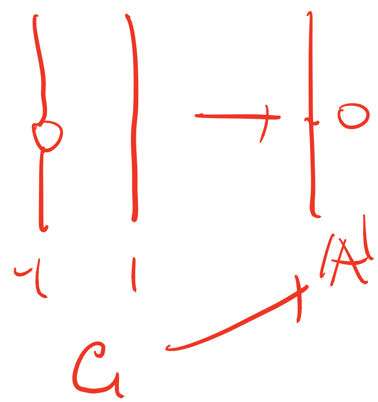
fixing everything except swaps origins

$[\mathcal{U}/(\mathbb{Z}/2)]$ 

$\parallel$

$\mathcal{X}$

$$\begin{tikzcd} \underline{\mathrm{Act}}(f)/\mathbb{A}^1 \arrow[r] & \mathbb{A}^1 \arrow[d] \arrow[dr, "f"] & \\ & \mathcal{U} \arrow[r] & \mathcal{X} \end{tikzcd}$$



not finite

$\mathcal{X} = B_{\mathbb{A}^1} G$

Ex: $B\mathbb{G}_m$ is not separated

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\
B\mathbb{G}_m \arrow[r] & B\mathbb{G}_m \times B\mathbb{G}_m
\end{tikzcd}$$

not proper

FACT If $\mathcal{X}$ with affine diagonal, then

$\mathcal{X}$ separated $\iff$ $\Delta_{\mathcal{X}}$ finite

Reason: proper + affine = finite

Alg stacks w/ affine diag & pos. dim'l stab. are not separated

Ex: $[\mathbb{A}^1/\mathbb{G}_m]$, $\mathrm{Bun}_{r,d}(C)$

Later: Show $\mathcal{M}_g$ is separated

Theorem (Val. Criteria for Univ. Closed/Proper/Separated). *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of algebraic stacks and consider a 2-commutative diagram*

$$
\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow_{\alpha}"] & \mathcal{X} \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r] & \mathcal{Y}
\end{tikzcd}
\qquad (\star)
$$

where R is a valuation ring with fraction field K. Then

(1) f is universally closed $\iff$ $\forall$ diagrams $(\star)$, $\exists$ an extension $R \to R'$ of valuation rings and $K \to K'$ of fraction fields together with a lifting

$$
\begin{tikzcd}
\operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\
\operatorname{Spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{Spec} R \arrow[r] & \mathcal{Y}.
\end{tikzcd}
$$

(2) f is separated $\iff$ any 2 liftings of $(\star)$ are isomorphic.

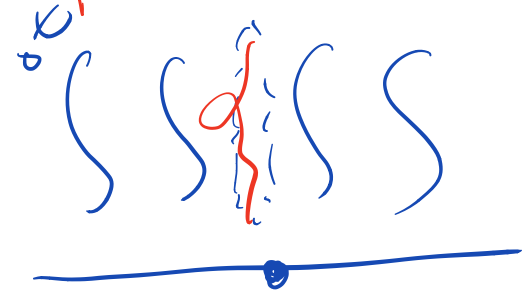
(3) f is proper $\iff$ every diagram $(\star)$ has a lifting after an extension $R \to R'$ and any 2 liftings are isomorphic.

Moreover, if $f\colon \mathcal{X} \to \mathcal{Y}$ is a finite type morphism of noetherian algebraic stacks, than it suffices to consider DVRs R and extensions such that $K \to K'$ is of finite transcendence degree.

Later $\mathcal{M}_g$ is proper by showing

$\forall$ DVRs R

$$
\begin{tikzcd}
 & C^{\times}_{K'} \arrow[r, hook, dashed, "\exists!"] \arrow[d] & C_{R'} \arrow[d, dashed] \\
C^{\times} \arrow[d] & \operatorname{Spec} K' \arrow[r] \arrow[dl] & \operatorname{Spec} R' \arrow[dl] \quad \exists \\
\operatorname{Spec} K \arrow[r, hook] & \operatorname{Spec} R &
\end{tikzcd}
$$



Ex: $B\mathbb{G}_m$

val crit for sep

$$
\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\text{trivial } \mathbb{G}_m\text{-torsor}"] \arrow[d] \arrow[dr, phantom, "\swarrow \text{trivial}"] & B\mathbb{G}_m \\
\operatorname{Spec} R \arrow[ur, bend left, "h_1 \text{ trivial}"] \arrow[ur, bend right, "h_2 \text{ trivial}"'] &
\end{tikzcd}
$$

π unif param $\in R$

$1/\pi \in$

Given isom $(h_1|_K \to h_2|_K) \in \mathbb{G}_m(K)$

Does not extend to $\mathbb{G}_m(R)$

§1. Quasi-coherent sheaves

Let $\mathcal{X}$ be a Deligne-Mumford stack.

Def The small étale site of $\mathcal{X}$ is category $\mathcal{X}_{ét}$ of schemes étale/$\mathcal{X}$.

object : $U \xrightarrow{ét} \mathcal{X}$

scheme

maps/$\mathcal{X}$

A covering $\{U_i \xrightarrow{ét} U\}$ s.t. $\coprod U_i \to U$

$\rightsquigarrow$ sheaves on $\mathcal{X}_{ét}$

$Sh(\mathcal{X}_{ét})$ = cat. of sheaves

$\mathcal{F}$ $\longleftarrow$ data $\forall\, U \xrightarrow{ét} \mathcal{X}$, U scheme

set of sections $\mathcal{F}(U \to \mathcal{X})$

Can extend to étale maps

$\mathcal{U} \to \mathcal{X}$ from DM stacks

$R \rightrightarrows U \xrightarrow{ét\ pres} \mathcal{U}$

Define

$$\mathcal{F}(\mathcal{U} \xrightarrow{ét} \mathcal{X}) = \mathrm{Eq}\left(\mathcal{F}(U \to \mathcal{X}) \rightrightarrows \mathcal{F}(R \to \mathcal{X})\right)$$

Can define <u>global sections</u>

$$\Gamma(\mathcal{X}, \mathcal{F}) = \mathcal{F}(\mathcal{X} \xrightarrow{id} \mathcal{X})$$

FACT For any map $\mathcal{X} \xrightarrow{f} \mathcal{Y}$

$\exists$ adjoint functors

$$Sh(\mathcal{X}_{ét}) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} Sh(\mathcal{Y}_{ét})$$

$\mathcal{F}$ $\qquad$ $\mathcal{G}$

DM

$$f_*\mathcal{F}(V \xrightarrow{ét} \mathcal{Y}) = \mathcal{F}(V \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X})$$

$$f^{-1}\mathcal{G}(U \xrightarrow{ét} \mathcal{X}) = \lim \mathcal{G}(V \to \mathcal{Y})$$

limit is over

$$\begin{array}{ccc} U & \to & V \\ {\scriptstyle ét}\downarrow & \swarrow & \downarrow{\scriptstyle ét} \\ \mathcal{X} & \to & \mathcal{Y} \end{array}$$

Structure sheaf on $\mathcal{X}$

$$\mathcal{O}_\mathcal{X}(U \xrightarrow{ét} \mathcal{X}) = \Gamma(U, \mathcal{O}_U)$$

sheaf of rings

$\rightsquigarrow$ notion $\mathcal{O}_\mathcal{X}$-modules

FACT $\mathcal{X} \to \mathcal{Y}$ map of DM stacks

$$\mathrm{Mod}(\mathcal{O}_\mathcal{X}) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{Mod}(\mathcal{O}_\mathcal{Y})$$

cat. of $\mathcal{O}_\mathcal{Y}$-mod

are adjoints

where $f^*(-) = f^{-1}(-) \otimes_{f^{-1}\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X}$

Def An $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$ is <u>quasi-coherent</u> if $\forall\, U \xrightarrow{ét} \mathcal{X}$ (scheme), the restriction $\mathcal{F}|_U$ to U_{zar} (Zariski top of U) is quasi-coherent.

FACT Let $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ map DM stacks

① f^* preserves quasi-coh

② If f is qcqs, f_* preserves quasi-coh.

Ex: G finite

$$\mathrm{QCoh}(BG) \cong \mathrm{Repn}(G) \ni V$$

W W

Consider $\operatorname{Spec} k \xrightarrow{p} BG \xrightarrow{\pi} \operatorname{Spec} k$

$p^*V = V$ forget G-action

$\pi_* V = V^G$ G-invariants

$\pi^* W = W$ trivial G-repn

$p_* W = W \otimes \underline{p_* k}$

$p_* k$ = reg. repn $\Gamma(G)$

Another perspective

$\mathcal{F} \in QCoh(\mathfrak{X})$

$\longleftrightarrow \forall S \to \mathfrak{X}$ not nec. ét

S scheme

q. coh sheaf $\mathcal{F}_S$ on S

s.t.
$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr] & & T \arrow[dl] \\ & \mathfrak{X} & \end{tikzcd} \qquad f^*\mathcal{F}_T \cong \mathcal{F}_S$$
canonical

Example Define $\mathcal{H} \in QCoh(\mathcal{M}_g)$

for $S \to \mathcal{M}_g \longleftrightarrow$
$$\begin{tikzcd} C \arrow[d, "\pi"', "sm"] \\ S \end{tikzcd}$$

$$\mathcal{H}_S := \pi_* \Omega_{C/S}$$

Hodge bundle

Other notions

- vect. bdls = loc. free of finite rank
  $\cup$
  line bdls
- coh. sheaves ($\mathfrak{X}$ noeth)
- $\mathcal{O}_{\mathfrak{X}}$-algebras $\mathcal{A}$
- rel. spectrum
  $\mathrm{Spec}\,\mathcal{A} \to \mathfrak{X}$
  affine

$$\begin{tikzcd} & & \mathcal{F} \\ \mathrm{Spec}\, k \arrow[r] & BG_x \arrow[r, hook] \arrow[ur, dash, dashed] & \mathfrak{X} \arrow[u, dash, dashed] \end{tikzcd}$$
$$G_x$$

§3. Local structure of DM stacks

$k = \bar{k}$

Theorem (Local Structure of DM Stacks).
Let $\mathcal{X}$ be a separated DM stack and $x \in \mathcal{X}(k)$ be a geom. point with stabilizer G_x. Then $\exists$ an affine, étale map

$$f \colon ([\operatorname{Spec} A/G_x], w) \xrightarrow{\text{ét}} (\mathcal{X}, x)$$

such that f induces an isom of stabilizer groups at w.

Here

$$\begin{tikzcd} w : \operatorname{Spec} k \arrow[r] \arrow[ddr, "x"'] & \operatorname{Spec} A \arrow[d] \\ & {[\operatorname{Spec} A/G_x] = \mathcal{W}} \arrow[d] \\ & \mathcal{X} \end{tikzcd}$$

$$\operatorname{Aut}_{\mathcal{W}(k)}(w) \xrightarrow{\sim} \operatorname{Aut}_{\mathcal{X}(k)}(x)$$

$$\Longrightarrow G_x \curvearrowright \operatorname{Spec} A \text{ fixes } w \in (\operatorname{Spec} A)(k)$$

Upshot: Tells us we can view DM stack as $[\operatorname{Spec} A/G]$ glued étale-locally

(G finite)

Notation

Let $\underset{\text{scheme}}{U} \xrightarrow{\text{ét}} \mathcal{X}$

$$(U/\mathcal{X})^d = \underbrace{U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U}_{d}$$

$S \to (U/\mathcal{X})^d \longleftrightarrow$ object $S \xrightarrow{x} \mathcal{X}$ & sections

$$\begin{tikzcd} U_S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ S \arrow[u, bend left, "s_1"] \arrow[u, bend right, "s_d"'] \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

Set $(U/\mathcal{X})^d_0 \subset (U/\mathcal{X})^d$ complement of all diag.

$S \to (U/\mathcal{X})^d_0 \longleftrightarrow$ object $S \xrightarrow{x} \mathcal{X}$ & disjoint sections

$$\begin{tikzcd} U_S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ S \arrow[u, bend left, "s_1"] \arrow[u, bend right, "s_d"'] \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

$$S_d \curvearrowright (U/\mathcal{X})^d_0 \subset (U/\mathcal{X})^d$$

Theorem (Local Structure of DM Stacks). *Let $\mathcal{X}$ be a separated DM stack and $x \in \mathcal{X}(k)$ be a geom. point with stabilizer G_x. Then $\exists$ an affine, étale map*

$$f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$$

such that f induces an isom of stabilizer groups at w.

Pf: Choose

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[r, "u"] \arrow[d, "\text{fin ét}", "\deg = d"'] & U \arrow[d, "\text{affine ét}"] \\ \operatorname{Spec} k \arrow[r] & BG_x \arrow[r] & \mathcal{X} \end{tikzcd}$$

affine

$d = |G_x|$ $(U/\mathcal{X})^d = U \times_{\mathcal{X}} \cdots U$ affine

$\cup$

$\exists\ W := (U/\mathcal{X})_0$ q. affine

$S \to (U/\mathcal{X})^d_0 \longleftrightarrow$ object $S \xrightarrow{x} \mathcal{X}$, z

$\cup$

W

disjoint sections $s_1, \ldots, s_d$

$$\begin{tikzcd} U_S \arrow[r] \arrow[d] & U \arrow[d] \\ S \arrow[r, "x"] \arrow[u, bend left, "s_1 \cdots s_d"] & \mathcal{X} \end{tikzcd}$$

Given $S \to (U/\mathcal{X})^d_0$

$$\begin{tikzcd} \coprod s_i(S) \arrow[r, hook, "cl"] \arrow[dr, "\text{fin ét}"', "\deg = d"] & U_S \arrow[r] \arrow[d] & U \arrow[d] \\ & S \arrow[r] & \mathcal{X} \end{tikzcd}$$

$S_d \curvearrowright (U/\mathcal{X})^d_0 = W$

$\downarrow$

$[W/S_d]$

$$\begin{tikzcd} S \arrow[r] & {[W/S_d]} & Z \arrow[l, leftrightarrow] \arrow[r, hook] \arrow[dr, "\text{fin ét}"', "\deg = d"] & U_S \arrow[r] \arrow[d] & U \arrow[d] \\ & & & S \arrow[r] & \mathcal{X} \end{tikzcd}$$

Have $w\colon \operatorname{Spec} k \to [W/S_d]$

$$\begin{tikzcd} Z = G_x \arrow[r] \arrow[d] & U \arrow[d] \\ \operatorname{Spec} k \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

Theorem (Local Structure of DM Stacks). *Let $\mathcal{X}$ be a separated DM stack and $x \in \mathcal{X}(k)$ be a geom. point with stabilizer G_x. Then $\exists$ an affine, étale map*

$$f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$$

such that f induces an isom of stabilizer groups at w.

$$S \to [W/S_d] \longleftarrow \begin{tikzcd} Z \arrow[r, hook] \arrow[dr, "\substack{\text{fin ét}\\ \deg = d}"'] & U_S \arrow[r, hook] \arrow[d] & U \arrow[d] \\ & S \arrow[r] & \mathcal{X} \end{tikzcd}$$

Have $w\colon \operatorname{Spec} k \to [W/S_d]$

$$\begin{tikzcd} Z = G_x \arrow[r] \arrow[d] \arrow[dr, phantom] & U \arrow[d] \\ \operatorname{Spec} k \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

Choice of ordering elements in G_x giving $\tilde{w}\colon \operatorname{Spec} k \to W$ scheme

Check: The stab of $\tilde{w}$ under S_d is $G_x \subset S_d$

PICTURE

$$\begin{tikzcd} S_d \curvearrowright (U/\mathcal{X})^d_0 \arrow[r, phantom, "\subset"] \arrow[dd, "\text{ét}"] & (U/\mathcal{X})^d \arrow[d, "\text{ét}"] \\ & U \arrow[d, "\text{ét}"] \\ {[W/S_d]} \arrow[r, "f"', "\text{ét}"] & \mathcal{X} \end{tikzcd}$$

$(U/\mathcal{X})^d_0 = W$

Check f is representable & étale $\Rightarrow$ f induces iso on stabilizers at w.

① Consider

$$[W/G_x] \to [W/S_d] \to \mathcal{X}$$

étale, preserves stab at w

② Know $w \in W$ a pt fixed by G_x. Choose $w \in W' \subset W$ affine ope

$w \in W'' = \bigcap g W'$ affine G-invariant

$W'' = \operatorname{Spec} A$

③ Why is it affine?

$$\begin{tikzcd}
U'' \arrow[r, "\text{fin}"] \arrow[d] & U' \arrow[r] \arrow[d] & U \arrow[d, "\text{ét}"] \\
\mathrm{Spec}\,A \arrow[r, "\text{finite}"] \arrow[rr, bend right, "\text{affine}"'] & {[\mathrm{Spec}\,A/G_x]} \arrow[r, "\text{repn}"] & \mathcal{X}
\end{tikzcd}$$

U'' aff. $\qquad U$ aff

Serre's crit $\Longrightarrow$ U' affine

$\Longrightarrow$ $[\mathrm{Spec}\,A/G_x] \to \mathcal{X}$ affine