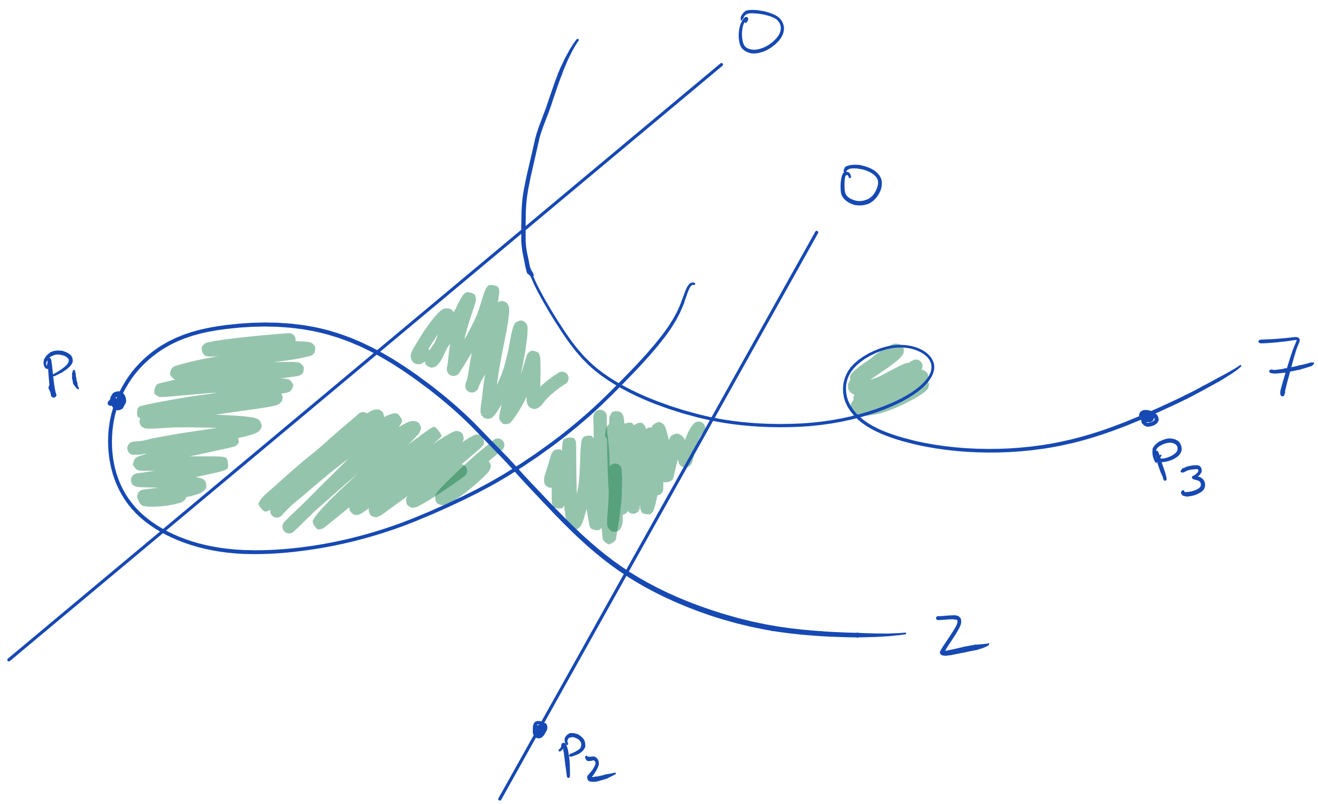
# LECTURE 14 : Stable curves




3-pointed stable curve of genus $g = (0+0+7+2)+5$
$= 14$

# §0. Review of nodes

**DEF** Let $C$ be a curve over $k$.

(1) If $k = \bar{k}$, $p \in C$ is a <u>node</u> if

$$\hat{\mathcal{O}}_{C,p} \cong k[[x,y]]/(xy)$$

(2) In general, $p \in C$ is a <u>node</u> if $\exists$ node $p' \in C_{\bar{k}}$ over $p$.

**Ex:** $0 \in C = \operatorname{Spec} \mathbb{R}[x,y]/(x^2+y^2)$ node

After $\mathbb{R} \to \mathbb{C}$, node becomes "<u>split</u>"

**Def** Say $C/k$ <u>nodal curve</u> if

$\forall p \in C$ either $p$ is smooth or a node.

**Theorem** (Local Structure of Nodes). *Let $\pi\colon \mathcal{C} \to S$ be a flat and fin. pres. map s.t. every fiber is a curve. If $p \in \mathcal{C}$ be a node in a fiber $\mathcal{C}_s$, there exists*

$$\begin{tikzcd}
(\mathcal{C}, p) \arrow[d] & (\mathcal{U}, u) \arrow[l, "\text{ét}"'] \arrow[r, "\text{ét}"] \arrow[d] & (\operatorname{Spec} A[x,y]/(xy - f), (s', 0)) \arrow[dl] \\
(S, s) & (\operatorname{Spec} A, s') \arrow[l, "\text{ét}"'] &
\end{tikzcd}$$

*where $f \in A$ is a function vanishing at $s'$.*

**Rmk:** $S = \operatorname{Spec} k$ Thm $\Rightarrow$

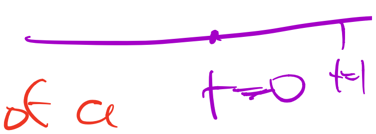
$\exists k \to k'$ fin. sep. & $p' \in C_{k'}$ s.t.

$$\hat{\mathcal{O}}_{C_{k'},p'} = k'[[x,y]]/(xy)$$

**Basic example**

$$\begin{tikzcd}
\operatorname{Spec} A[x,y]/(xy-f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \mathbb{Z}[x,y,t]/(xy-t) \arrow[d] \\
\operatorname{Spec} A \arrow[r] & \operatorname{Spec} \mathbb{Z}[t]
\end{tikzcd}$$

$f \mapsfrom t$


<u>Thm says:</u> every def. of a node étale-locally is the pullback of this example

**Theorem** (Local Structure of Nodes)**.** *Let $\pi\colon \mathcal{C} \to S$ be a flat and fin. pres. map s.t. every fiber is a curve. If $p \in \mathcal{C}$ be a node in a fiber $\mathcal{C}_s$, there exists*

$$\begin{tikzcd}
(\mathcal{C}, p) \arrow[d] & (\mathcal{U}, u) \arrow[l, "\text{ét}"'] \arrow[r, "\text{ét}"] \arrow[d] & (\operatorname{Spec} A[x,y]/(xy - f), (s', 0)) \arrow[dl] \\
(S, s) & (\operatorname{Spec} A, s') \arrow[l, "\text{ét}"'] &
\end{tikzcd}$$

*where $f \in A$ is a function vanishing at $s'$.*

Variant when $\mathcal{C}_s$ is non-reduced and $p \in (\mathcal{C}_s)_{red}$ is nodal

Exer Let $R$ be a DVR; $t \in R$ uniformizer

Suppose $p \in \mathcal{C}$ regular

$$\begin{tikzcd}
p \arrow[d, mapsto] \in \mathcal{C} \arrow[d, "\substack{\text{flat} \\ \text{fin. pres}}"] \\
(t) = 0 \in \operatorname{Spec} R
\end{tikzcd}$$

$\exists\, R \to R'$ étale

① If $p \in (\mathcal{C}_0)_{red}$ smooth

$$\exists \operatorname{Spec} R'[x,y]/(x^a - t) \xrightarrow{\text{ét}} \mathcal{C}$$
$$0 \longmapsto p$$

② If $p \in (\mathcal{C}_0)_{red}$ node

$$\exists \operatorname{Spec} R'[x,y]/(x^a y^b - t) \xrightarrow{\text{ét}} \mathcal{C}$$
$$0 \longmapsto p$$

# Application

**Theorem** (Local Structure of Nodes). *Let $\pi\colon \mathcal{C} \to S$ be a flat and fin. pres. map s.t. every fiber is a curve. If $p \in \mathcal{C}$ be a node in a fiber $\mathcal{C}_s$, there exists*

$$\begin{tikzcd}
(\mathcal{C}, p) \arrow[d] & (\mathcal{U}, u) \arrow[l, "\text{ét}"', "g"] \arrow[r, "\text{ét}"] \arrow[d] & (\operatorname{Spec} A[x,y]/(xy - f), (s', 0)) \arrow[dl] \\
(S, s) & (\operatorname{Spec} A, s') \arrow[l, "\text{ét}"'] &
\end{tikzcd}$$

*where $f \in A$ is a function vanishing at $s'$.*

- More than we need
- Conceptual understanding of nodes & their defs

<u>Cor</u>: Let $\pi : \mathcal{C} \to S$ as in thm.

Then $\mathcal{C}^{\leq \text{node}} = \{ p \in \mathcal{C} \mid p \in \mathcal{C}_{\pi(p)}$ is smooth or node$\}$

$\subset \mathcal{C}$ open.

<u>Pf</u>: Know smooth locus is open

If $p \in \mathcal{C}_s$ node $(s = \pi(p))$

$\Rightarrow p \in g(\mathcal{U}) \subset \mathcal{C}^{\leq \text{node}}$ open

<u>Cor</u>: If in addition $\pi : \mathcal{C} \to S$ proper,

then $S^{\leq \text{node}} := \{ s \in S \mid \mathcal{C}_s \text{ nodal} \}$

$\subseteq S$ open

<u>Pf</u>:

$$S^{\leq \text{node}} = S \setminus \overbrace{\pi(\underbrace{\mathcal{C} \setminus \mathcal{C}^{\leq \text{node}}}_{\text{closed locus in } \mathcal{C} \text{ of non-nodes}})}^{\pi \text{ proper} \Rightarrow \text{closed}}$$

**Cor**: If in addition $\pi : C \to S$ proper, then $S^{\le \text{node}} := \{ s \in S \mid C_s \text{ nodal} \} \subseteq S$ open

We will apply this result next time to show that the stack

$\mathcal{M}_g^{\le \text{nodal}}$ with objects $\left( \begin{array}{c} C \\ \downarrow \text{flat, fin pres, proper} \\ S \end{array} \text{ s.t. } \forall s \in S,\ C_s \text{ nodal} \right)$

$\downarrow$

$Sch$

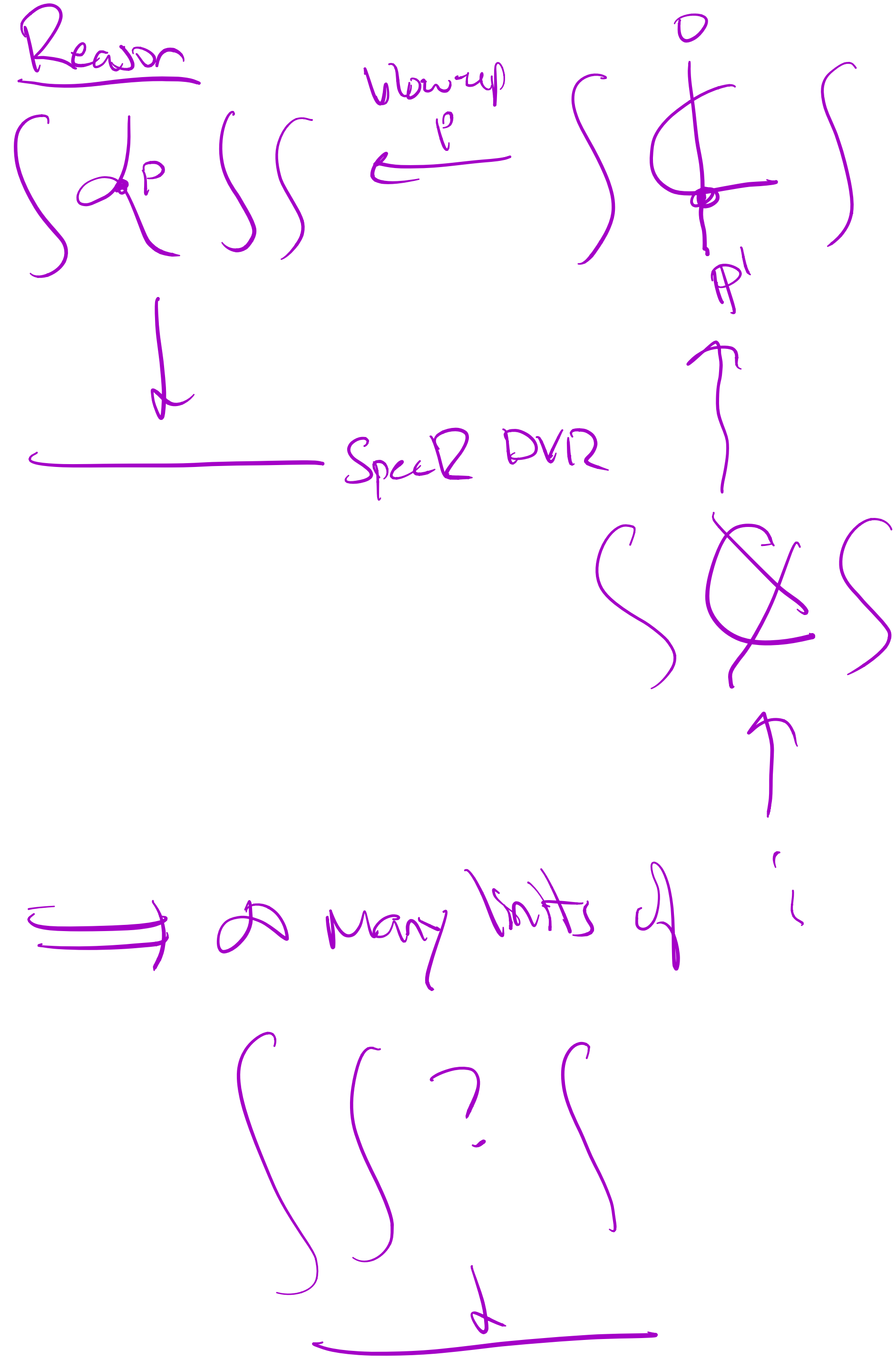
is algebraic.

Problems $\mathcal{M}_g^{\le \text{nodal}}$

① not separated

② not bounded (i.e. not f.type)

## Reason



# §1 Stable curves

Mumford & Meyer

**Defn** An n-pointed curve $C$ over $k$ is a curve $C$ & an ordered set of points $p_1, \dots, p_n \in C(k)$

**Notation:** Say $q \in C$ is special if $q$ is marked or a node

**Def** (Stable curves)**.** An $n$-pointed curve $(C, p_1, \dots, p_n)$ over $k$ is *stable* if $C$ is a geom connected, nodal and projective curve, and $p_1, \dots, p_n \in C$ are distinct smooth points such that

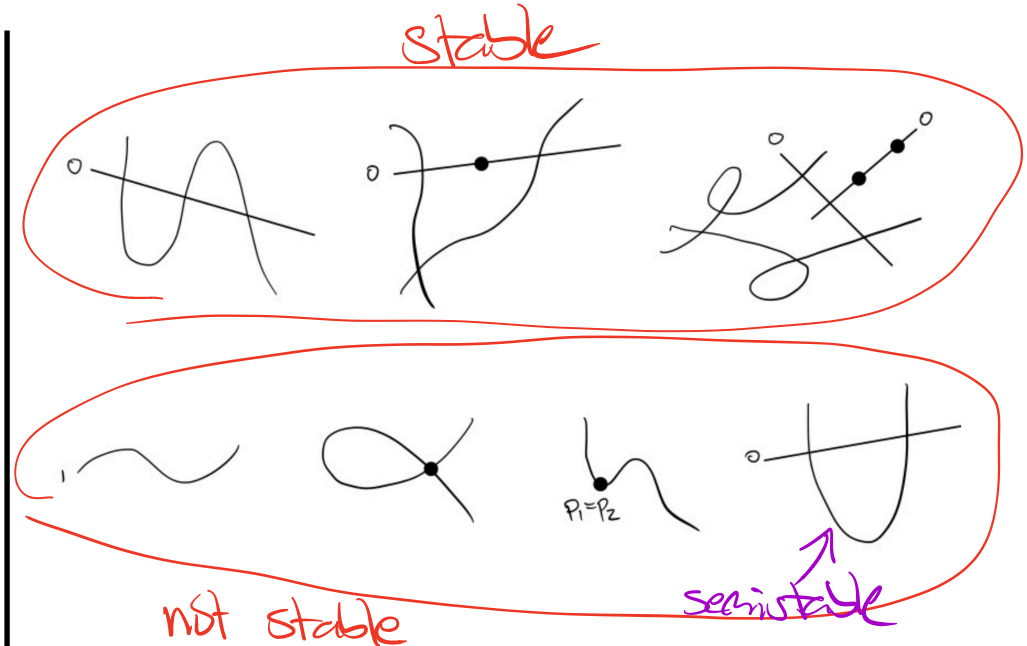
(1) every smooth rational subcurve $\mathbb{P}^1 \subset C$ contains at least 3 special points, and

(2) $C$ is not of genus 1 without marked points.

semistable: replace "3" with "2" in condition (1)

prestable: drop (1) & (2)

"nodal + distinct, smooth marked points"




**Rmk:** $\nexists$ stable curves if $(g,n) = (0,0), (0,1), (0,2)$ or $(1,0)$

$\iff 2g-2+n \leq 0$

Sometime impose $2g-2+n > 0$

Let $(C,p_1,\dots,p_n)$ n-pointed prestable

Take normalization

$$\begin{array}{ll} \tilde{C} & \pi^{-1}(p_i)=\{\tilde{p}_i\} \\ \big\downarrow \pi & \pi^{-1}(C^{sing})=\{\underbrace{\tilde{q}_1,\dots,\tilde{q}_m}_{\text{all pts in } \tilde{C} \text{ over nodes}}\} \\ C & \end{array}$$

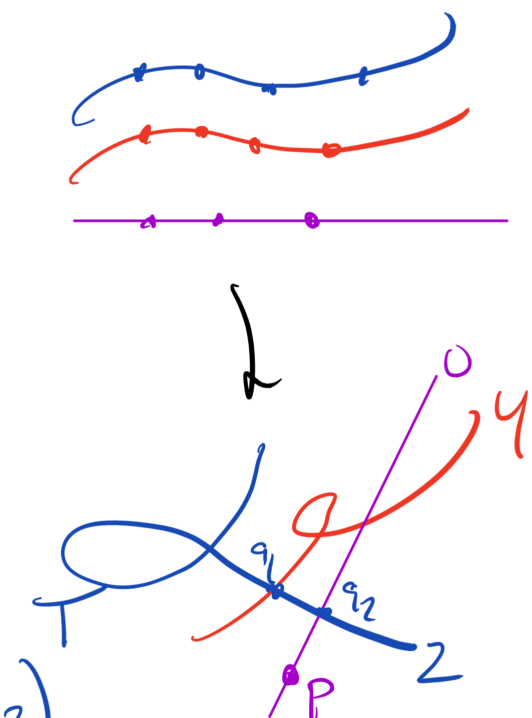
same is true for ampleness of $\omega_C(p_1+\dots+p_n)$

<u>Exer</u> Show $(C,\{p_i\})$ stable

$\Longleftrightarrow$ every conn component of $\underbrace{(\tilde{C},\{\tilde{p}_i\},\{\tilde{q}_i\})}_{\text{pted normalization}}$ is stable

<u>Example</u>

$$\begin{array}{c} \tilde{C} \\ \text{finite} \; \big\downarrow \pi \\ C \end{array}$$




$\omega_C|_T=\omega_T(q_1+q_2)$

<u>Fact</u> The only smooth n-pted curves $(C,p_1,\dots,p_n)$ with $|\underbrace{\mathrm{Aut}(C,\{p_i\})}_{\alpha: C \xrightarrow{\sim} C \text{ w/ } \alpha(p_i)=p_i}|=\infty$ are

$$\left[\begin{array}{l} \bullet\ C=\mathbb{P}^1 \quad n=0,1,2 \\ \bullet\ C=\text{genus } 1 \text{ and } n=0 \end{array}\right] \quad 2g-2+n\le 0$$

<u>Prop</u> Let $(C,\{p_i\})$ n-pted prestable. TFAE

① $(C,\{p_i\})$ stable

② $\mathrm{Aut}(C,\{p_i\})$ finite

③ $\omega_C(p_1+\dots+p_n)$ ample

<u>Pf:</u> ① $\Leftrightarrow$ ② follow from Exer & Fact

① $\Leftrightarrow$ ③ : Use fact: if $C$ nodal curve and $T\subset C$ subcurve, $\omega_C|_T=\omega_T(T\cap T^c)$

Therefore

$\omega_C(p_1+\dots+p_n)$ ample $\Leftrightarrow$ $\pi^*(\omega_C(p_1+\dots+p_n))$ ample

$\Leftrightarrow \forall T\subset\tilde{C}$ $\omega_C(p_1+\dots+p_n)|_T = \omega_T(\sum_{p_i\in T} p_i + T\cap T^c)$

$\Leftrightarrow \forall T\subset\tilde{C}$ $(T, \sum_{p_i\in T} p_i + T\cap T^c)$ stable

Exer: If $(C, \{p_i\})$ is stable, then for $k \geq 3$

$$(\omega_C(p_1 + \cdots + p_n))^{\otimes k} \text{ very ample}$$

Hint Assume $n=0$ for simplicity

Need to show $\omega_C^{\otimes k}$ separates points & tangent vectors. That is,

(a) $\forall x, y \in C$

$$H^0(C, \omega_C^{\otimes k}) \to (\omega_C^{\otimes k} \otimes k(x)) \oplus (\omega_C^{\otimes k} \otimes k(y))$$

(b) $\forall x \in C$

$$H^0(C, \omega_C^{\otimes k}) \to \omega_C^{\otimes k} \otimes \mathcal{O}_{C,x}/m_x^2$$

Both (a) & (b) come from

$$0 \to \omega_C^{\otimes k} \otimes m_x m_y \to \omega_C^{\otimes k} \to \omega_C^{\otimes k} \otimes \mathcal{O}_C/m_x m_y \to 0$$

Suffices $H^1(C, \omega_C^{\otimes k} \otimes m_x m_y) = 0$

$\forall x, y \in C$ possibly equal $\quad \| \text{SD}$

$$\text{Hom}(m_x m_y, \omega_C^{\otimes k})$$

Show vanishing using case analysis of $x, y \in C$ nodes or smooth

# §2. Families of stable curves

**Definition** (Families).

(1) A *family of $n$-pointed nodal curves* is a flat, proper and finitely presented morphism $\mathcal{C} \to S$ of schemes with $n$ sections $\sigma_1, \ldots, \sigma_n \colon S \to \mathcal{C}$ such that every geometric fiber is a (reduced) connected nodal curve.

(2) A *family of $n$-pointed stable curves* is a family $\mathcal{C} \to S$ of $n$-pointed nodal curves such that every geometric fiber $(\mathcal{C}_s, \sigma_1(s), \ldots, \sigma_n(s))$ is stable.

(3) same semistable, prestable

Define $\overline{\mathcal{M}}_{g,n}$ prestack

objects $\overline{\mathcal{M}}_{g,n}$ are $\left\{ \begin{tikzcd} \mathcal{C} \arrow[d] \\ S \arrow[u, bend left, "\sigma_1"] \arrow[u, bend left=60, "\sigma_n \cdots"] \end{tikzcd} \text{ stable } n\text{-pted families} \right\}$

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n} \arrow[d] & S \arrow[d, mapsto] \\ Sch & S \end{tikzcd}$$

morphism = cart. diagrams preserving the sections

## Rel. dualizing sheaf

Fact: If $\mathcal{C} \to S$ prestable, then

$\mathcal{C} \to S$ loc. complete intersection

$\Longrightarrow \exists$ rel. dual. sheaf $\omega_{\mathcal{C}/S}$

Ref: Hartshorne Res & Duality

Liu Alg geom & arith curves

Property:

$$\begin{tikzcd} \mathcal{C}_T \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ T \arrow[r] & S \end{tikzcd} \qquad \omega_{\mathcal{C}/S}\big|_{\mathcal{C}_T} = \omega_{\mathcal{C}_T/T}$$

In part, $\omega_{\mathcal{C}/S}\big|_{\mathcal{C}_s} = \omega_{\mathcal{C}_s/k(s)}$

**Proposition** (Properties of Families of Stable Curves). *Let $(\mathcal{C} \to S, \{\sigma_i\})$ be a family of $n$-pointed stable curves of genus $g$, and set $L := \omega_{\mathcal{C}/S}(\sum_i \sigma_i)$. If $k \geq 3$, then $L^{\otimes k}$ is relatively very ample and $\pi_* L^{\otimes k}$ is a vector bundle of rank $(2k-1)(g-1) + kn$.*

Reason

Checked on fibers

coh & base change

RR

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d] & \mathbb{P}(\pi_* L^{\otimes k}) \arrow[dl] \\ S & \end{tikzcd}$$

**Proposition** (Openness of Stability). *Let $(\mathcal{C} \to S, \{\sigma_i\})$ be a family of $n$-pointed nodal curves. The locus of points $s \in S$ such that $(\mathcal{C}_s, \{\sigma_i(s)\})$ is stable is open.*

Pf: The locus $s \in S$ where $\sigma_1(s), \dots, \sigma_n(s)$ distinct & smooth is open

$\implies$ Assume $(\mathcal{C} \to S, \{\sigma_i\})$ prestable

Two arguments

① $\underline{\mathrm{Aut}}(\mathcal{C}/S, \sigma_1, \dots, \sigma_n) \underset{e}{\overset{\text{f.type group scheme}}{\rightleftarrows}} S$

$\implies S \to \mathbb{Z}$

$s \mapsto \dim \mathrm{Aut}(\mathcal{C}_s, \{\sigma_i(s)\})$

upper semi-cont

$\implies \{s \in S \mid (\mathcal{C}_s, \{\sigma_i(s)\}) \text{ stable}\}$

$= \{s \in S \mid \dim \mathrm{Aut}(\mathcal{C}_s, \{\sigma_i(s)\}) = 0\}$

open

② $\mathcal{C} \ni L = \omega_{\mathcal{C}/S}(\sigma_1 + \dots + \sigma_n)$

$\mathcal{C} \to S$

$\{s \in S \mid L_s \text{ ample on } \mathcal{C}_s\}$

$= \{s \in S \mid (\mathcal{C}_s, \{\sigma_i(s)\}) \text{ stable}\}$

open

## §3. Automorphisms, deformations & obstructions

Auts, Defs & Obs of a stable curve $C$ are governed by $\mathrm{Ext}^i(\Omega_C, \mathcal{O}_C)$ for $i=0,1,2$

**Proposition.** *Let $(C, p_1, \ldots, p_n)$ be an n-pointed stable curve of genus $g$ over $k$. Then*

$$\dim_k \mathrm{Ext}^i(\Omega_C(\sum_i p_i), \mathcal{O}_C) = \begin{cases} 0 & \text{if } i=0 \\ 3g-3+n & \text{if } i=1 \\ 0 & \text{if } i=2 \end{cases}$$

Later: we will apply this

$\mathrm{Ext}^0 = 0 \Rightarrow \overline{\mathcal{M}}_{g,n}$ DM

$\mathrm{Ext}^2 = 0 \Rightarrow \overline{\mathcal{M}}_{g,n}$ smooth

$\mathrm{Ext}^1 \Rightarrow \dim \overline{\mathcal{M}}_{g,n}$ over field $= 3g-3+n$

Pf ($n=0$ case)

$(\tilde{C}, \tilde{\Sigma})$ pointed normalization

$\underline{i=0}$:

$$\begin{array}{ccc} \tilde{\Sigma} = \pi^{-1}(\Sigma) & \subset & \tilde{C} \\ \downarrow & & \downarrow \pi \\ \{\text{nodes}\} = \Sigma & \subset & C \end{array}$$

normalization

Claim: $\mathrm{Hom}(\Omega_C, \mathcal{O}_C) = \mathrm{Hom}(\Omega_{\tilde{C}}(\tilde{\Sigma}), \mathcal{O}_{\tilde{C}})$

line-bdl

reg. vect. fields on $C \leftrightarrow$ reg. vect. fields on $\tilde{C}$ vanishing at preimages of nodes

Skip

Claim $\Rightarrow$

Since $(\tilde{C}, \tilde{\Sigma})$ stable

$\Rightarrow H^0(T_{\tilde{C}}(-\tilde{\Sigma})) = 0$

deg < 0

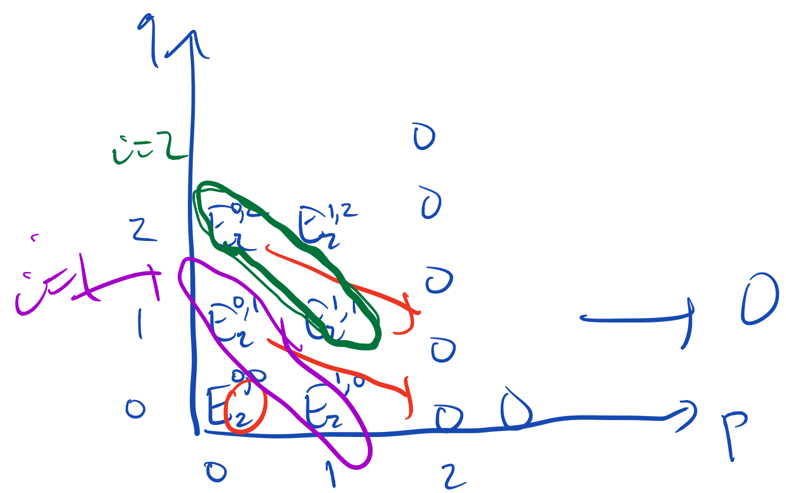
**Proposition.** *Let $(C, p_1, \dots, p_n)$ be an $n$-pointed stable curve of genus $g$ over $k$. Then*

$$\dim_k \mathrm{Ext}^i(\Omega_C(\sum_i p_i), \mathcal{O}_C) = \begin{cases} 0 & \text{if } i = 0 \\ 3g-3+n & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}$$

$\underline{i=2}$ Use local-to-global spectral seq

$$E_2^{p,q} = H^p(C, \mathcal{E}xt^q(\Omega_C, \mathcal{O}_C)) \Rightarrow \mathrm{Ext}^{p+q}(\Omega_C, \mathcal{O}_C)$$

Since $\dim C = 1$, $E_2^{p,q} = 0$ $p > 1$




$$E_2^{1,1} = H^1(C, \underbrace{\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)}_{\text{dim 0 support}}) = 0$$

b/c $\Omega_C$ line bdl away from nodes

$z \in C$ $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_z = \mathrm{Ext}^1(\Omega_{C,z}, \mathcal{O}_{C,z})$

$z \in C$ smooth $= 0$

$$E_2^{0,2} = H^0(\mathcal{E}xt^2(\Omega_C, \mathcal{O}_C))$$

Since $C$ is loc. complete inter.

$\Rightarrow \exists$ locally free res.

$$0 \to E_1 \to E_0 \to \Omega_C \to 0$$

Explicitly, $p \in U \overset{C}{\hookrightarrow} \mathbb{A}^n$ def $I$

$$0 \to I/I^2 \to \Omega_{\mathbb{A}^n}|_U \to \Omega_U$$

$\Rightarrow \mathcal{E}xt^2(\Omega_C, \mathcal{O}_C) = 0$

$\Rightarrow E_2^{0,2} = 0$

$\Rightarrow \mathrm{Ext}^2(\Omega_C, \mathcal{O}_C) = 0$

**Proposition.** *Let $(C, p_1, \dots, p_n)$ be an $n$-pointed stable curve of genus $g$ over $k$. Then*

$$\dim_k \mathrm{Ext}^i(\Omega_C(\sum_i p_i), \mathcal{O}_C) = \begin{cases} 0 & \text{if } i = 0 \\ 3g-3+n & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}$$

Setup

$$\begin{array}{ccc} \tilde{\Sigma} = \pi^{-1}(\Sigma) & \subset & \tilde{C} \quad \text{normalization} \\ \downarrow & & \downarrow \pi \\ \{\text{nodes}\} = \Sigma & \subset & C \end{array}$$

$i=1$ Low degree exact sequence of spectral sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow \underbrace{\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)}_{\text{1st order defs of } C} \longrightarrow E_2^{0,1} \longrightarrow \cancel{E_2^{2,0}} = 0$$

$$E_2^{1,0} = H^1(\mathcal{H}om(\Omega_C, \mathcal{O}_C)) = \mathrm{Ext}^1(\Omega_{\tilde{C}}[\tilde{\Sigma}], \mathcal{O}_{\tilde{C}})$$

1st order def of C preserve the node

=

1st order defs of pted norm

$$\left[\begin{array}{ccc} C & \hookrightarrow & C_\varepsilon \\ \downarrow & \square & \downarrow \text{flat} \\ \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} k[\varepsilon] \end{array}\right]$$

$$E_2^{0,1} = H^0(\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)) = \prod_{z \in \Sigma} \underbrace{\mathrm{Ext}^1(\hat{\Omega}_{C,z}, \hat{\mathcal{O}}_{C,z})}_{\text{1st order defs of } \hat{\mathcal{O}}_{C,z}}$$

dim 0 supp

$= \Omega_{\hat{\mathcal{O}}_{C,z}}$

$$\left[\begin{array}{ccc} \operatorname{Spec} \hat{\mathcal{O}}_{C,z} & \hookrightarrow & \operatorname{Spec} \hat{\mathcal{O}}_{C_\varepsilon,z} \\ \downarrow & \square & \downarrow \\ \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} k[\varepsilon] \end{array}\right]$$

<u>Upshot</u>: Short exact sequence

line bdl

$$0 \to \underbrace{\mathrm{Ext}^1(\Omega_{\tilde C}(\tilde\Sigma), \mathcal{O}_{\tilde C})} \to \underbrace{\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)}_{\text{Goal}} \to \underbrace{\prod_{z\in\Sigma} \overbrace{\mathrm{Ext}^1(\Omega_{\hat{\mathcal{O}}_{C,p}}, \hat{\mathcal{O}}_{C,p})}^{\text{1-dim'l local calc.}}}_{\dim = \#\text{nodes}} \to 0$$

$\tilde C = \bigsqcup \tilde C_i$

$\tilde\Sigma_i = \tilde\Sigma \cap \tilde C_i$

$$\dim = \sum_i \mathrm{Ext}^1(\Omega_{\tilde C_i}(\tilde\Sigma_i), \mathcal{O}_{\tilde C_i})$$

$$= \sum_i h^1(T_{\tilde C_i}(-\tilde\Sigma_i))$$

$$\overset{SD}{=} \sum_i h^0(\Omega^{\otimes 2}_{\tilde C_i}(\tilde\Sigma_i))$$

$$\overset{RR}{=} \sum_i \left(\deg\left(\Omega^{\otimes 2}_{\tilde C_i}(\tilde\Sigma_i)\right) + 1 - \tilde g_i\right)$$

genus of $\tilde C_i$

$$= \sum_i \left(3\tilde g_i - 3 + |\tilde\Sigma_i|\right) = 3\sum_i \tilde g_i - 3\,\#\text{comp} + 2\,\#\text{nodes}$$

$$\dim = 3\sum \tilde g_i - 3\,\#\text{comp} + 3\,\#\text{nodes}$$

$$= \boxed{3g-3} \quad \text{b/c}$$

$$g = \sum \tilde g_i - \#\text{comp} + \#\text{nodes} + 1$$