Lecture 15 : Stable reduction

Schedule

- Stable reduction ($1\frac{1}{3}$ lectures)
  - $\overline{\mathcal{M}}_{g,n}$ proper

  Today

- Gluing & forgetful morphisms ($\frac{2}{3}$ lecture)
  - $\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2-2}$
  - $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g+1,n-2}$
  - $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ univ family

  next Monday March 8

- Irreducibility (1 lecture)
  - $\overline{\mathcal{M}}_{g,n}$ irreducible

  next Wed March 10

- Projectivity (1 lecture)
  - The coarse moduli space $\overline{M}_{g,n}$ is projective.

  Monday, Mar 15

§0. Recap

Theorem. *If $2g-2+n>0$, then $\overline{\mathrm{M}}_{g,n}$ is a quasi-compact Deligne–Mumford stack smooth over* $\operatorname{Spec}\mathbb{Z}$ *of relative dimension $3g-3+n$.*

Proposition. *Let $(C,p_1,\dots,p_n)$ be an n-pointed stable curve of genus g over k. Then*

$$\dim_k \operatorname{Ext}^i\Big(\Omega_C\big(\sum_i p_i\big),\mathcal{O}_C\Big)=\begin{cases}0 & \text{if } i=0\\ 3g-3+n & \text{if } i=1\\ 0 & \text{if } i=2\end{cases}$$

Proof

① $\overline{\mathcal{M}}_{g,n}$ is algebraic & loc. of f.type/$\mathbb{Z}$

Last time: the stack $\mathcal{M}_{g,n}^{\mathrm{all}}$ of all curves is alg & loc. of f.type/$\mathbb{Z}$

$$\overline{\mathcal{M}}_{g,n} \subset \mathcal{M}_{g,n}^{\mathrm{all}} \text{ open}$$

② $\overline{\mathcal{M}}_{g,n}$ is DM

- For a stable curve $(C,p_1,\dots,p_n)$, $\operatorname{Aut}(C,\{p_i\})$ is (abstract) finite group
- Know

$$T_e\,\underline{\operatorname{Aut}}(C,\{p_i\}) = \operatorname{Ext}^0\big(\Omega_C(\textstyle\sum p_i),\mathcal{O}_C\big) = 0$$

$\Rightarrow$ $\underline{\operatorname{Aut}}(C,\{p_i\})$ finite & reduced

By Equiv. Characterization of DM stacks $\Rightarrow$ $\overline{\mathcal{M}}_{g,n}$ DM

③ $\overline{\mathcal{M}}_{g,n} \to \operatorname{Spec}\mathbb{Z}$ is smooth

Use Formal Lifting Criterion

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[d, hook] \arrow[dr, "{[C,\{p_i\}]}"] & \\
\operatorname{Spec} A_0 \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n} \arrow[d] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}\mathbb{Z}
\end{tikzcd}$$

$A \twoheadrightarrow A_0$ of local artinian rings with $\ker(A\twoheadrightarrow A_0)=k$

$\exists\, ob \in \operatorname{Ext}^2(\Omega_C(\sum p_i),\mathcal{O}_C) = 0$ s.t. $ob=0 \iff \exists$ dotted arrow

④ For a field k, $\dim(\overline{\mathcal{M}}_{g,n}\times_{\mathbb{Z}} k) = 3g-3+n$

Use def theory [extensions where $A=k[\varepsilon]\twoheadrightarrow A_0=k$]

$$T_{\overline{\mathcal{M}}_{g,n}\times_{\mathbb{Z}} k,\,[C,\{p_i\}]} = \operatorname{Ext}^1\big(\Omega_C(p_1+\dots+p_n),\mathcal{O}_C\big)$$

Theorem. *If $2g-2+n>0$, then $\overline{\mathrm{M}}_{g,n}$ is a quasi-compact Deligne–Mumford stack smooth over* $\operatorname{Spec}\mathbb{Z}$ *of relative dimension $3g-3+n$.*

⑤ Boundedness : $\overline{M}_{g,n}$ quasi-compact

Use fact : If $(C,\{p_i\})$ stable,

then $(\omega_C(p_1+\cdots+p_n))^{\otimes 3} = L$ very ample

$$C \overset{|L|}{\hookrightarrow} \mathbb{P}^N \qquad N = h^0(L)$$

$p_1,\dots,p_n$

Let $P(t)$ be Hilb. poly &

quasi-compact projective

$$H \subset \mathrm{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}) \times (\mathbb{P}^N_{\mathbb{Z}})^n$$

$$\{(C \hookrightarrow \mathbb{P}^N, p_1,\dots,p_n) \mid p_i \in C\}$$

quasi-compact

$$\begin{array}{ccc} H' & \subset & H \\ \downarrow & & \downarrow f \\ \overline{M}_{g,n} & \subset & M^{all}_{g,n} \end{array} \qquad \begin{array}{c} (C \hookrightarrow \mathbb{P}^N, p_i) \\ \downarrow \\ (C, p_i) \end{array}$$

Know $f(H) \supset \overline{M}_{g,n}$

$\Longrightarrow \overline{M}_{g,n}$ quasi-compact

§1. Overview of stable reduction

GOAL $\overline{\mathcal{M}}_{g,n} \longrightarrow \operatorname{Spec} \mathbb{Z}$ proper

Theorem (Valuation Criterion for Properness). *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of noetherian algebraic stacks. Then f is proper if and only if for every DVR R with fraction field K and 2-commutative diagram*

$$\begin{tikzcd} \Delta^\times = \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \Delta = \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd} \qquad (\star)$$

Frac(R)

DVR

such that

(1) there exists an extension $R \to R'$ of DVRs (with $K \to K'$ of fraction fields) together with a lifting

existence

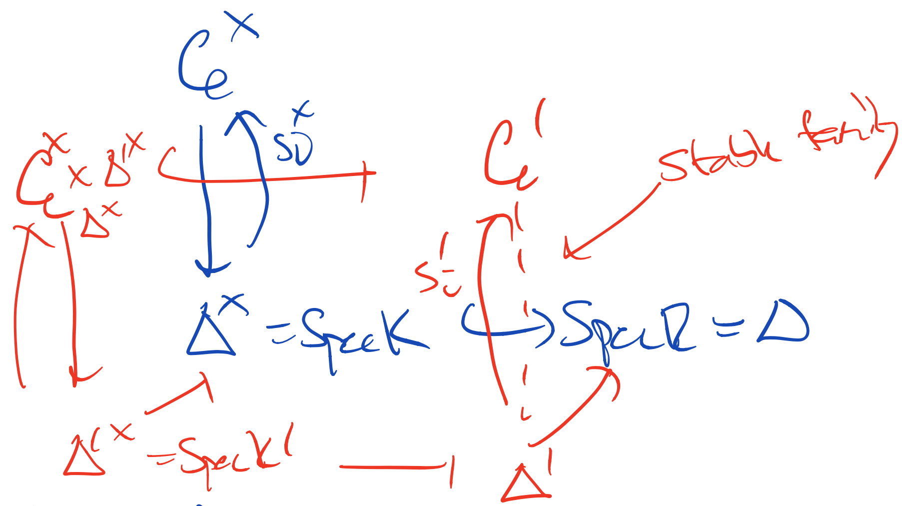
$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{Spec} R \arrow[r] & \mathcal{Y} \end{tikzcd}$$

(2) any two liftings of $(\star)$ are isomorphic.

uniqueness

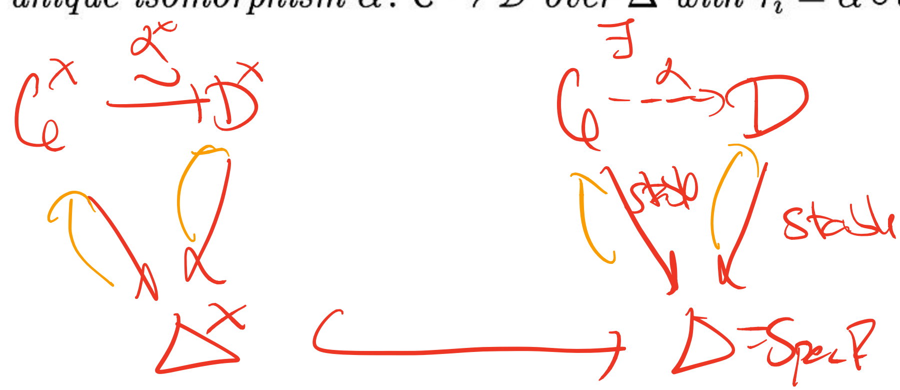
Existence

Theorem (Stable Reduction). *If $(\mathcal{C}^* \to \Delta^*, s_1^*, \ldots, s_n^*)$ is a family of n-pointed stable curves of genus g, then there exists a finite cover $\Delta' \to \Delta$ of spectrums of DVRs and a family $(\mathcal{C}' \to \Delta', s_1', \ldots, s_n')$ of stable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$.*



Uniqueness

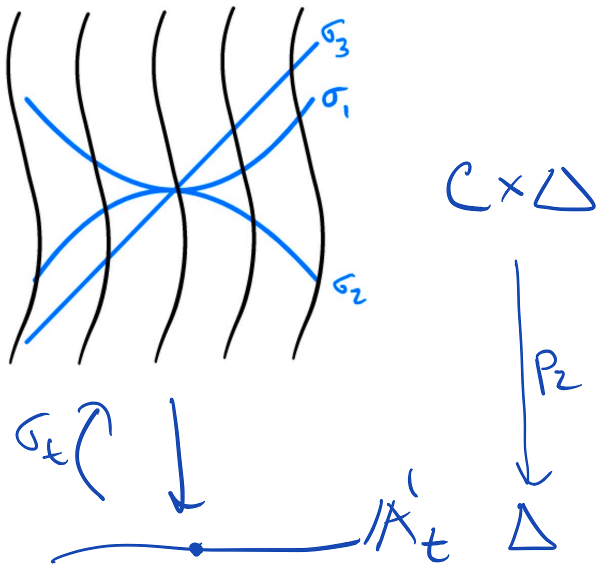
Proposition (Separatedness). *If $(\mathcal{C} \to \Delta, \sigma_1, \ldots, \sigma_n)$ and $(\mathcal{D} \to \Delta, \tau_1, \ldots, \tau_n)$ are families of n-pointed stable curves, then any isomorphism $\alpha^*\colon \mathcal{C}^* \to \mathcal{D}^*$ over Δ^* with $\tau_i^* = \alpha^* \circ \sigma_i^*$ of the generic fibers extends to a unique isomorphism $\alpha\colon \mathcal{C} \to \mathcal{D}$ over Δ with $\tau_i = \alpha \circ \sigma_i$.*



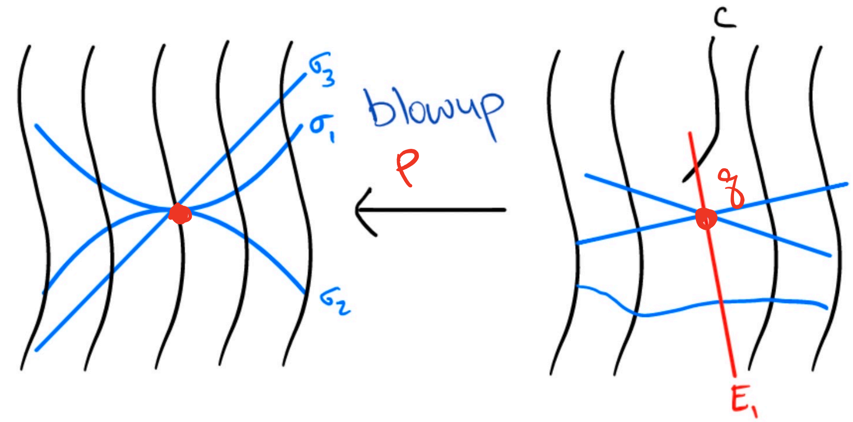
§2. First examples

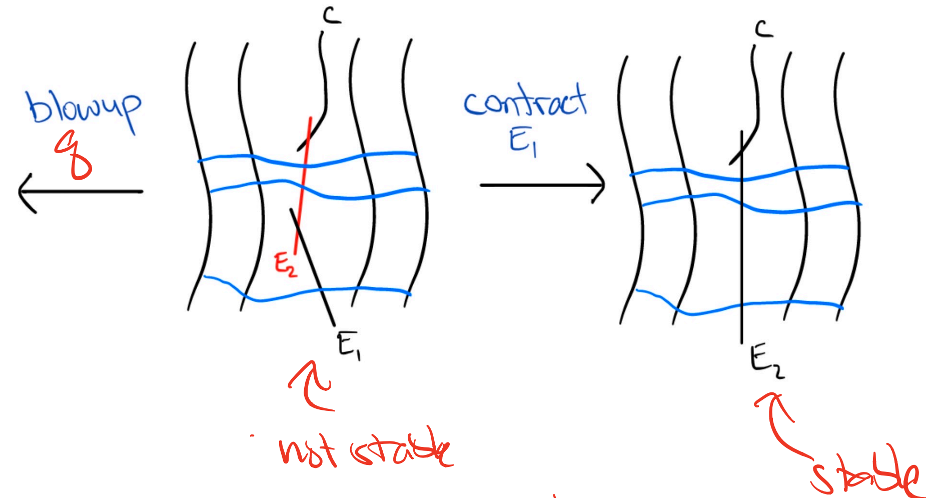
Example 1 (colliding marked points)

Let C be smooth, proj & conn curve $/k$



Étale locally: $t \longmapsto (\underset{\sigma_1}{\underset{\parallel}{t^2}}, \underset{\sigma_2}{\underset{\parallel}{-t^2}}, \underset{\sigma_3}{\underset{\parallel}{4t}})$



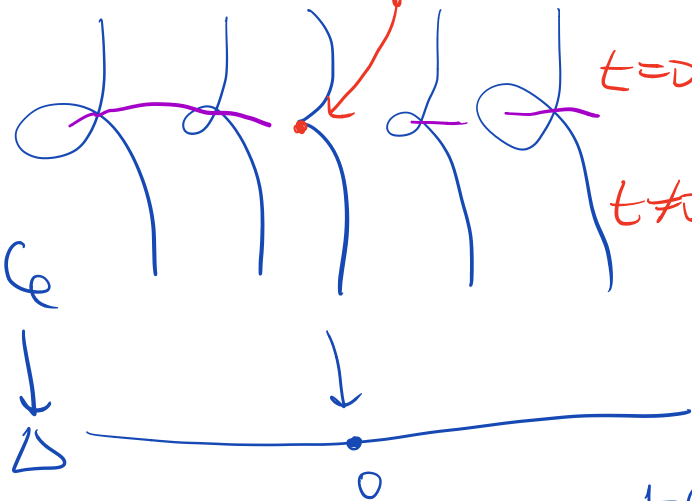


E_1 is a rational bridge

Example 2 (node degenerating to a cusp)

Suppose we have a family

Local eqn: $y^2 = x^3 + tx^2$



$t=0$: $y^2 = x^3$ (cusp)

$t \neq 0$ $y^2 = x^2$ (node)

$\Delta = \operatorname{Spec} R$

$t \in R$ uniformizer

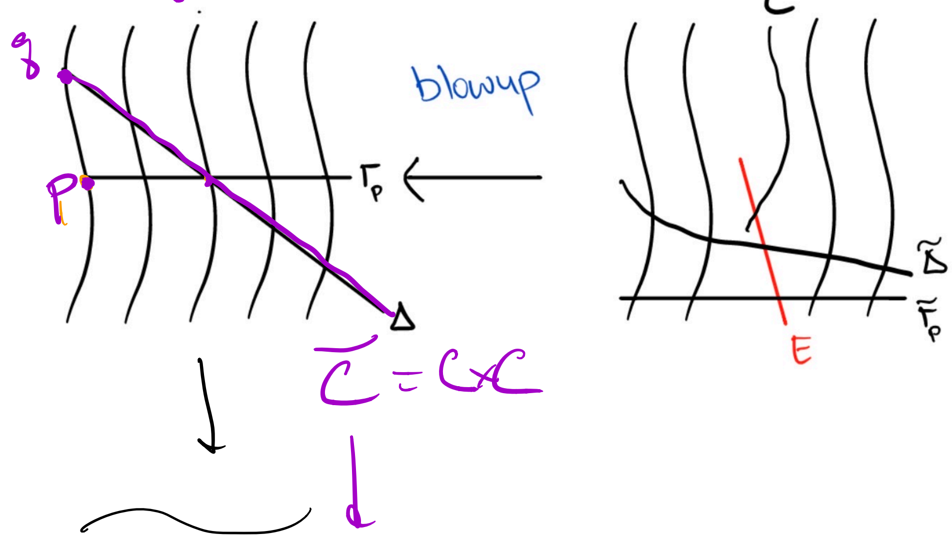
Ques: What is the stable limit?

Let $\Delta^\times \longrightarrow \mathcal{C}^\times$

$$\begin{array}{ccc} \tilde{\mathcal{C}}_e^{\times} & \longrightarrow & \tilde{\mathcal{C}} \\ {\scriptstyle s_1, s_2} \uparrow\downarrow & \searrow \quad \swarrow & \\ & \Delta^\times & \end{array}$$

pointed normalization

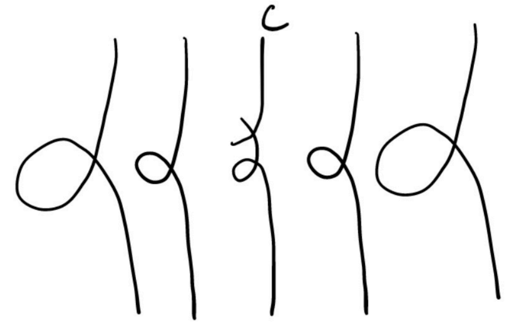
Sliding $q \to p$



$\tilde{C} = C \times C$

$\Delta = C$

Glue $\tilde{\Delta}$ & $\tilde{\Gamma}_p$ $\longrightarrow$



Note! Neither example is generically smooth

$y^2 = x^3 + t$

more difficult

§3. Stable reduction: basic strategy

Reference: Harris & Morrison

Given

$$\begin{tikzcd} \mathcal{C}_e^\times \arrow[r, hook] \arrow[d, "\text{stable curve}"] & \mathcal{C}_e \supset \mathcal{C}_0 \arrow[d, "\text{flat}"] \\ \Delta^\times = \operatorname{Spec} K \arrow[r, hook] & \Delta = \operatorname{Spec} R \ni 0 \end{tikzcd}$$

STEP 1 Find some extension $\mathcal{C}_e \xrightarrow{\text{flat}} \Delta$ with possibly very singular $\mathcal{C}_0$

$$\begin{tikzcd} & \mathbb{P}^n_K \arrow[r, hook] & \mathbb{P}^n_R \\ \mathcal{C}_e^\times \arrow[ur, "|\omega^{\otimes 3}_{\mathcal{C}^\times}|"] \arrow[rr, hook] \arrow[d] & & \mathcal{C} := \overline{\mathcal{C}_e^\times} \arrow[u, hook] \arrow[d, "\text{flat}"] \\ \operatorname{Spec} K = \Delta^\times \arrow[rr, hook] & & \Delta = \operatorname{Spec} R \end{tikzcd}$$

flat (all assoc pt live over gen. pt)

STEP 2 Reduce to case where generic fiber $\mathcal{C}_e^\times \to \Delta^\times$ is smooth

Idea: If $\mathcal{C}_e^\times$ has k nodes take pointed normalization $(\widetilde{\mathcal{C}}^\times, \tilde{p}_1, \dots, \tilde{p}_{2k})$

- Perform stable reduction to each component
- Then take nodal union of sections

[For induction to work, need pointed case) For simplicity, assume $n=0$

STEP 3 Use Embedded Resolutions

$$\begin{tikzcd} \exists\ \widetilde{\mathcal{C}}_e \arrow[rr, "\text{proj}"', "\text{brtl}"] \arrow[dr] & & \mathcal{C}_e \arrow[dl, "\text{gen smooth}"] \\ & \Delta & \end{tikzcd}$$

regular

finite sequence of blowups at closed pts of $\mathcal{C}_0$

s.t. $\widetilde{\mathcal{C}}_0$ set-theoretic normal crossings

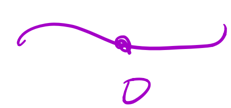
(i.e. $(\widetilde{\mathcal{C}}_0)_{red}$ nodal)

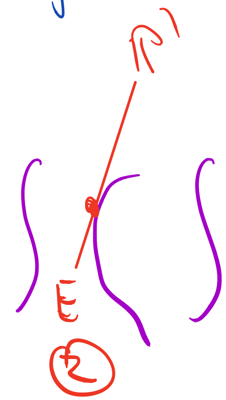
Example Local eqn $y^2 = x^3 + t$

$$\begin{tikzcd} \mathcal{C} \arrow[d] & \operatorname{Bl}_p \mathcal{C} \arrow[l, "\text{blowup}"'] \arrow[dl] \\ \mathbb{A}^1_t & \end{tikzcd}$$

$(\tilde{x}, \tilde{x}\tilde{y}) \leftarrow\!\shortmid (\tilde{x}, \tilde{y})$ (1 chart)

$\tilde{x} = x$, $\tilde{x}\tilde{y} = y$





Central fiber of $\operatorname{Bl}_p \mathcal{C} \to \mathbb{A}^1$

$=$ preimage of $y^2 - x^3$

$= \tilde{x}^2\tilde{y}^2 - \tilde{x}^3 = \underline{\tilde{x}^2}(\tilde{y}^2 - \tilde{x})$

2 copies of exc. div

Given
$$\begin{tikzcd}
 & & \mathcal{C}_0 \arrow[dl] \arrow[d] \\
\mathcal{C}^\times \arrow[r, hook] \arrow[d, "\text{stable curve}"] & \mathcal{C} \arrow[d, "\text{flat proper}"] & 0 \arrow[dl, hook'] \\
\Delta^\times = \operatorname{Spec} K \arrow[r, hook] & \Delta = \operatorname{Spec} R &
\end{tikzcd}$$

__Step 2__ Find some extension $\mathcal{C} \xrightarrow{\text{flat}} \Delta$.

__Step 1__ Reduce to case where generic fiber $\mathcal{C}^\times \to \Delta^\times$ is smooth

__Step 3__ Use Embedded Resolutions

$$\begin{tikzcd}
\exists\ \tilde{\mathcal{C}} \arrow[rr, "\text{proj}"', "\text{bir'l}"] \arrow[dr] & & \mathcal{C} \arrow[dl] \\
 & \Delta &
\end{tikzcd}$$

regular ($\tilde{\mathcal{C}}$)

finite sequence of blowups at closed pts of $\mathcal{C}_0$

s.t. $\tilde{\mathcal{C}}_0$ set-theoretic normal crossings

(i.e. $(\tilde{\mathcal{C}}_0)_{red}$ nodal curve)

__Step 4__ Take ramified base extension

$\Delta' = \operatorname{Spec} R' \to \Delta = \operatorname{Spec} R$, $t \mapsto t^m$

s.t. central fiber of the normalization $\widetilde{\mathcal{C} \times_\Delta \Delta'}$ is reduced & nodal.

__Step 5__ Take minimal resolution & Contract rational tail & bridges

I'll give more details on (4) & (5)

§4. Birational geometry of surfaces

Principle: Understanding the moduli of dim n varieties & specifically stable reduction requires birtl geometry & minimal model program in dim n+1

Surface = integral scheme of f.type/$k=\bar{k}$ of dim = 2

References: Hartshorne Ch. V
Kollár, Lectures on Resolutions of Singularities

Theorem (Embedded Resolutions). *Let X be a surface and $X_0 \subset X$ be a curve. There is a projective birational morphism $\widetilde{X} \to X$ obtained as a finite sequence of blow-ups at reduced points of X_0 yielding such that $\widetilde{X}$ is smooth and the preimage $\widetilde{X}_0$ of X_0 has set-theoretic normal crossings, i.e. $(\widetilde{X}_0)_{\mathrm{red}}$ is nodal.*

Theorem (Minimal Resolutions). *Let X be a surface. There exists a unique projective birational morphism $\widetilde{X} \to X$ from a smooth surface such that every other resolution $Y \to X$ factors as $Y \to \widetilde{X} \to X$.*

Theorem (Castelnuovo's Contraction Theorem). *Let X be a smooth projective surface and E a smooth rational curve with $E^2 = -1$. Then there is a projective birational morphism $X \to Y$ to a smooth projective surface and a point $y \in Y$ such that $X_y = E$ and $X \setminus E \to Y \setminus \{y\}$ is an isomorphism.*

Process of contracting −1 curves terminates.

Corollary (Existence of Relative Minimal Models). *A smooth surface X admits a projective birational morphism $X \to X_{\mathrm{min}}$ to a smooth surface such that every projective birational morphism $X_{\mathrm{min}} \to Y$ to a smooth surface is an isomorphism. In particular X_{min} has no smooth rational -1 curves.*

§5. Stable reduction (char=0 proof)

Assume R is a DVR over $\mathbb{Q}$.

Given

$$\begin{tikzcd} \mathcal{C}^\times \arrow[d, "\text{stable curve}"] & \subset & \mathcal{C}^{\text{reg.}} \arrow[d, "\text{flat}"] \\ \Delta^\times = \operatorname{Spec} K \arrow[rr, hook] & & \Delta = \operatorname{Spec} R \end{tikzcd}$$

STEPS 1–3 Reduce to $\mathcal{C}^\times \to \Delta^\times$ smooth, find some limit & apply Embedded Resolutions

$\rightsquigarrow$ Given $\mathcal{C} \to \Delta$ with $(\mathcal{C}_0)_{red}$ nodal and $\mathcal{C}$ regular

STEP 4 Take ramified base extension

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta' = \operatorname{Spec} R \arrow[r] & \Delta = \operatorname{Spec} R \end{tikzcd}, \quad t \mapsto t^m$$

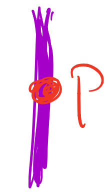
s.t. central fiber of the normalization $\widetilde{\mathcal{C}'}$ is reduced & nodal.

Take m = lowest common multiple of multiplicities of components

Recall: We have an étale-local description of $\mathcal{C} \to \Delta$ near $p \in \mathcal{C}_0$

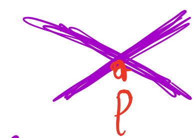
(1) $p \in (\mathcal{C}_0)_{red}$ smooth

$$(x,y) \mapsto x^a$$



(2) $p \in (\mathcal{C}_0)_{red}$ nodal

$$(x,y) \mapsto x^a y^b$$



For (1), $p \in \mathcal{C}$ has local eqn

$$x^a - t$$

$\exists !$, unique preimage $p' \in \mathcal{C}' = \mathcal{C} \times_\Delta \Delta'$ & has local eqn

CHAR =0

$$\left\{ x^a - t^m = \prod_{i=0}^{a-1} \left(x - \rho^i t^{m/a}\right) \right.$$

reduced

ρ = primitive a^{th} root of unity $\quad \boxed{\Rightarrow \bar{p}' \in \widetilde{\mathcal{C}}'_0}$

Take $\widetilde{\mathcal{C}}' \to \mathcal{C}'$ normalization

$\bar{p}' \mapsto p'$ local eqn $x - \rho^i t^{m/a}$

For (2), $p \in (C_0)_{red}$ node

$(x,y) \mapsto x^a y^b$

$$\boxed{\begin{array}{l}\text{Exer: Under } \tilde{C}' \to C_e \\ \text{each preimage } \tilde{p}' \text{ of } p \\ \text{has local eqn } t^k = xy\end{array}}$$

$\Rightarrow \tilde{p}' \in \tilde{C}_0$ nodal & reduced

$\Rightarrow \tilde{p}' \in \tilde{C}_e'$ A_{k-1}-sing

Upshot! $\tilde{C}_e' \to \Delta$

Replace C_e with $\tilde{C}'$, we get

$C_e \to \Delta$ nodal family

singularities

STEP 5 Take minimal resolution and contract rational tails & bridges

Replace C_e with a minimal resolution

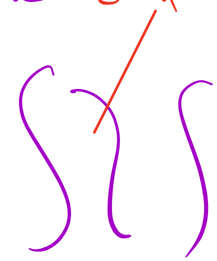
$\rightsquigarrow C_e \to \Delta$ prestable family

reg

Take: stable model by contracting rtl tails & bridges

$$\begin{tikzcd} C_e \arrow[r] \arrow[dr] & C_e^{st} \arrow[d, "\text{stable}"] \\ & \Delta \end{tikzcd}$$

$E \cong \mathbb{P}^1$



$E^2 = -1$

Explicitly,

- Contract rational tails

Contract E by Castelnuovo

$$\begin{tikzcd} C_e \arrow[r] \arrow[dr] & C_{e,min} \arrow[d] \\ & \Delta \end{tikzcd}$$

rel. min model

regular

semistable family (Semistable reduction)

- Contract rational bridges

§6. Summary ($\Delta = \operatorname{Spec} R$, R DVR)

In characteristic 0, we have proved

Theorem (Semistable Reduction). *If $\mathcal{C}^* \to \Delta^* = \operatorname{Spec} K$ is a smooth, proj and geom conn curve, there exists a cover $\Delta' \to \Delta$ of spectrums of DVRs and a family $\mathcal{C}' \to \Delta'$ of semistable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$ such that $\mathcal{C}'$ is regular.*

Theorem (Stable Reduction). *If $(\mathcal{C}^* \to \Delta^*, s_1^*, \ldots, s_n^*)$ is a family of n-pointed stable curves of genus g, then there exists a finite cover $\Delta' \to \Delta$ of spectrums of DVRs and a family $(\mathcal{C}' \to \Delta', s_1', \ldots, s_n')$ of stable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$.*

History: Deligne-Mumford ('69) showed stable reduction in positive/mixed char by using stable reduction for ab. varieties.

- Artin-Winters ('71)

Gave a general proof (pos/mixed char) along the same lines as us:

Vague idea: Step 1-3 give

$(\mathcal{C}_e)_{\text{central}}$ nodal

$$\begin{tikzcd} \mathcal{C}^\times_{K'} \arrow[r, hook] \arrow[d] & \mathcal{C}^\times_e \arrow[r, hook] \arrow[d, "\text{smooth}"] & \mathcal{C}_e \arrow[d, "\text{flat}"] \\ \operatorname{Spec} K' \arrow[r] & \Delta^\times = \operatorname{Spec} K \arrow[r, hook] & \Delta = \operatorname{Spec} R \end{tikzcd}$$

Take $K \to K'$ field ext

- $\mathcal{C}^\times_{K'}(K') \neq \emptyset$ $\quad \ell \gg 0$
- $\operatorname{Pic}(\mathcal{C}^\times_{K'})[\ell] = (\mathbb{Z}/\ell\mathbb{Z})^{2g}$

Take $R' \subset K'$ DVR

Show central fibre of $\mathcal{C}_{R'}$ is reduced & nodal