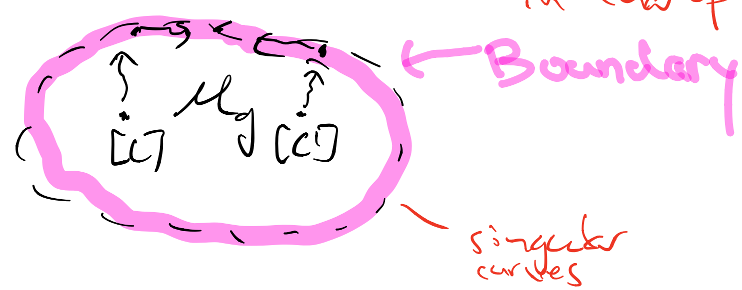
Lecture 17: Irreducibility

> **Thm.** *The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper and irreducible Deligne–Mumford stack of dimension $3g-3$ which admits a projective coarse moduli space.*

We know everything but irreducibility & projectivity.

Caveat: We only proved properness in char $=0$

Today: We'll use properness$/\mathbb{Z}$ $\Longrightarrow$ irreducibility in char $=p$



Today's Outline

1. Background on branched covers

2. Clebsch-Hurwitz argument (1872 & 1891) — char $=0$

3. Fulton's appendix to Harris & Mumford's paper "On the Kodaira Dimension of M_g" (1982) — char $=0$ & completely algebraic

Exploit compactification

4.
   - Deligne-Mumford's 2 arguments in (1969) "On the irreducibility of M_g"
   - Fulton's argument in "Hurwitz schemes and irreducibility of M_g" (1969)

   char p: reduce to char 0 — $p \gg 0$

§0. The goal

THM $\overline{\mathcal{M}}_{g,n}$ is irreducible

Remark 1 As $\overline{\mathcal{M}}_{g,n}$ is smooth, this is equivalent to

$\iff$ $\overline{\mathcal{M}}_{g,n}$ connected

$\iff$ $\overline{\mathcal{M}}_{g}$ connected

$\iff$ $\mathcal{M}_{g}$ connected & dense in $\overline{\mathcal{M}}_{g}$

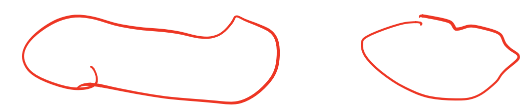
Remark 2 We have $\overline{\mathcal{M}}_{g,n} \xrightarrow{\text{cms}} \overline{M}_{g,n}$ and $|\mathcal{M}_{g,n}| = |\overline{M}_{g,n}|$ as top. spaces.

$\rightsquigarrow$ statements on stack are equiv. to statements on cms

Why do we care?

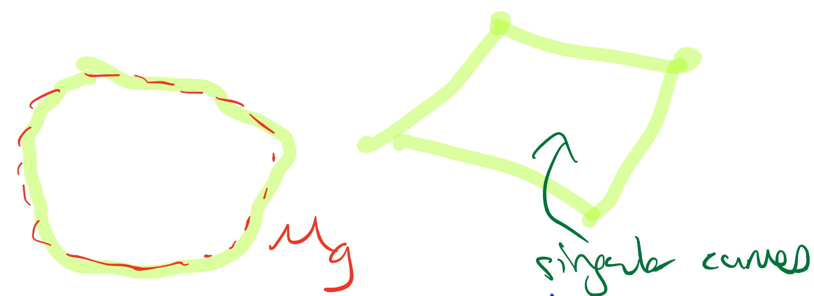
- $\mathcal{M}_g$ connected $\iff$ Genus is the only discrete invariant

Rules out $\mathcal{M}_g$ looking like



- $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ dense $\iff$ $\nexists$ component of $\overline{\mathcal{M}}_g$ consisting of entirely singular curves

Rules out



- We cover algebraic approaches!

There are other topological/analytic arguments (e.g. using Teichmüller)

§1. Background on branched covers

DEF A branched covering of $\mathbb{P}^1$ is a morphism $f: C \xrightarrow{\text{finite}} \mathbb{P}^1$ $d = \deg f$

smooth & conn

with $K(\mathbb{P}^1) \to K(C)$ separable.

Rmk: f is étale at points p where $(\Omega_{C/\mathbb{P}^1})_p = 0$

- Say f is ramified at p of index e if $\text{length}(\Omega_{C/\mathbb{P}^1})_p = e-1$

Example: $X = \mathbb{A}^1 \to \mathbb{A}^1 = Y \quad x \mapsto x^d$

$$\Omega_{X/Y} = k[x]\langle dx\rangle / (d x^{d-1} dx)$$

char $\nmid d \Rightarrow \text{length}(\Omega_{X/Y})_0 = d-1$

- The ramification divisor is

$$R = \sum_{p \in C} (\Omega_{C/\mathbb{P}^1})_p \cdot p$$

The short exact sequence

$$0 \to f^*\Omega_{\mathbb{P}^1} \to \Omega_C \to \Omega_{C/\mathbb{P}^1} \to 0$$

implies $K_C = f^*K_{\mathbb{P}^1} + R$ as divisors

Take degrees

Riemann-Hurwitz $d = \deg(f)$

$$2g-2 = d(-2) + R$$

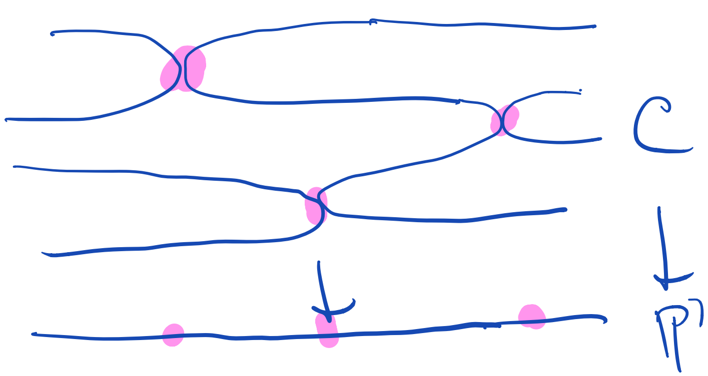
$$\Rightarrow \boxed{\deg R = 2d + 2g - 2}$$

DEF A branched covering $C \longrightarrow \mathbb{P}^1$ is <u>simply branched</u> if

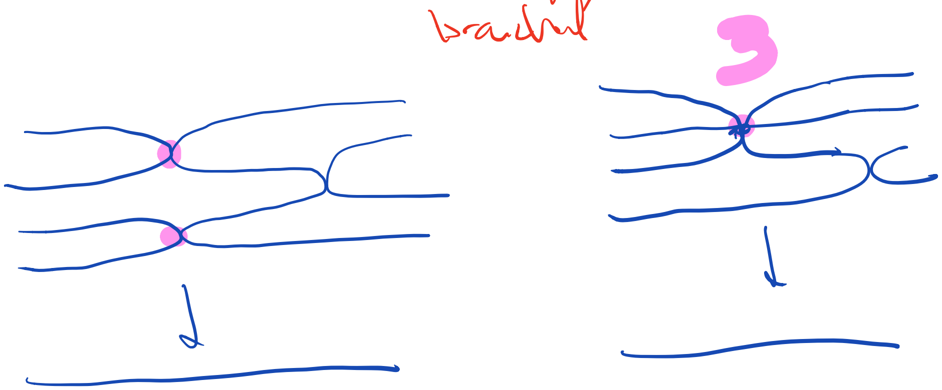
(1) every ramification point has index 2

(2) $\exists$ at most one ramification point in every fiber

simply branched



not simply branched



Riemann-Hurwitz $\Rightarrow$ a simple branched covering $C \xrightarrow{d} \mathbb{P}^1$ is ramified over $b := 2d + 2g - 2$ distinct points in $\mathbb{P}^1$

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char = 0

Lemma 1 Let C be a smooth, proj, conn curve of genus g & L a line bdl of degree $d >> 0$. Then for a general $V \subset H^0(L)$ of dim 2, $C \xrightarrow{V} \mathbb{P}^1$ is simply branched

RR

Proof Dimension count: $h^0(L) = d + 1 - g$

$$\dim Gr(2, H^0(L)) = 2(h^0(L) - 2)$$
$$= 2(d - g - 1)$$

- If $C \xrightarrow{V} \mathbb{P}^1$ is not simply branched, then either (a) V has a base pt

  *(b) $\exists$ ram. pt with index > 2

  (c) $\exists$ 2 ram. pts in same fiber

<u>Case (b)</u>

$\exists s \in V$ s.t. $s \in H^0(L(-3p))$ for $p \in C$

$$\dim \{ V \in Gr(2, H^0(L)) \mid C \xrightarrow{V} \mathbb{P}^1 \text{ satisfies (b)} \}$$
$$= \dim \mathbb{P} H^0(L(-3p)) + \dim \mathbb{P}(H^0(L)/s) + 1$$
$$= d - 3 + (1 - g) - 1 + d - g - 1 + 1$$
$$= 2d - 2g - 3 < 2(d - g - 1)$$

Lemma 2 If $C \to \mathbb{P}^1$ is a simply branched cover of degree $d > 2$, then $\mathrm{Aut}(C/\mathbb{P}^1) = \{1\}$

Reason: Any $\alpha : C \to C$ over $\mathbb{P}^1$ would fix the $2d+2g-2$ branched pts

(Mayer) $\nexists$ non-trivial auts fixing $2g+2$ points

Define space (either as top. space or alg space)

$$H_{d,b} := \{ C \xrightarrow{d} \mathbb{P}^1 \text{ simply branched over } b \text{ points} \}$$

$$\boxed{b = 2g+2d-2}$$

We have maps

$[C \to \mathbb{P}^1] \mapsto [C]$, $\mapsto$ b branched points

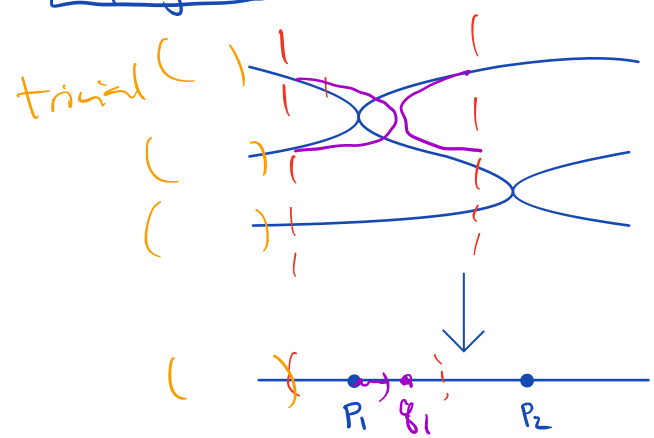
$$H_{d,b} \to M_g, \qquad H_{d,b} \to \underbrace{\mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta}_{\text{unordered distinct pts}} = \mathbb{P}^b \setminus \Delta$$

Lemma 3 In char $= 0$, $H_{d,b} \to \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$ is finite & étale, i.e. a covering space.

$\Longrightarrow$ Lemma implies that any $C \to \mathbb{P}^1$ can be deformed so that the branched locus is general

Pf of étaleness (skip finiteness)

• $\boxed{\text{Topological}}$ Given $C \to \mathbb{P}^1$



Map on tangent spaces of $H_{d,b} \to M_g$

• $\boxed{\text{Algebraic}}$ $\mathrm{Def}^{1st}(C \xrightarrow{f} \mathbb{P}^1) \to \underbrace{\mathrm{Def}^{1st}(\{p_i\}_{i=1}^b \subset \mathbb{P}^1)}_{b \text{ dim'l}}$

$\qquad \| \; H^0(C, N_f)$

where $0 \to T_C \to f^* T_{\mathbb{P}^1} \to N_f \to 0$

$$0 \to \underbrace{H^0(f^* T_{\mathbb{P}^1})}_{2d+1-g} \to \underbrace{H^0(N_f)}_{b} \to \underbrace{H^1(T_C)}_{3g-3} \to 0$$

RR $\Rightarrow$ $2d+1-g$

§2: Clebsch–Hurwitz proof / $\mathbb{C}$

References
- Clebsch (1872)
- Hurwitz (1891)
- Fulton "Hurwitz schemes and irreducibility..." (1969)

Need non-algebraic input:

Riemann Existence Thm: There are bijections

$$\left\{ C \to \mathbb{P}^1 \text{ algebraic branched coverings} \right\} \longleftrightarrow \left\{ C \to \mathbb{P}^1 \text{ topological branched coverings} \right\}$$

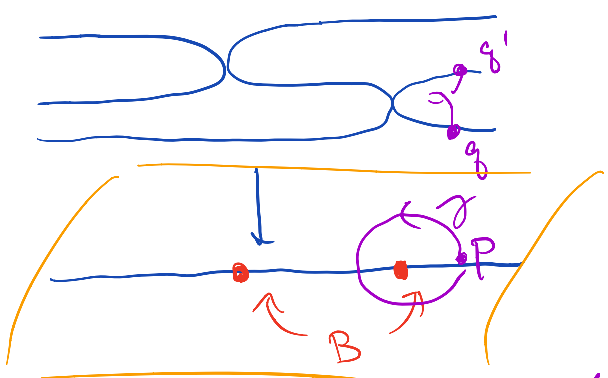
$$\updownarrow$$

hard

$$\longleftrightarrow \left\{ C \to \mathbb{P}^1 \text{ holomorphic branched covering} \right\}$$

Need monodromy action

Given $C \to \mathbb{P}^1$, let $B \subset \mathbb{P}^1$ ramification locus



$$\begin{array}{c} C \\ \downarrow f \\ \mathbb{P}^1_{\mathbb{C}} \end{array}$$

$\gamma \in \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus B, p)$

Tracing q under the lifting of path γ to C gives another pt $q' \in \pi^{-1}(p)$ ← d elements

$\rightsquigarrow \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus B, p) \curvearrowright f^{-1}(p)$ $\quad \gamma \cdot q = q'$

$\rightsquigarrow \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus B, p) \xrightarrow{\rho} S_d$ gp hom

$\quad \| $

$\langle \sigma_1, \dots, \sigma_b \mid \sigma_1 \cdots \sigma_b = 1 \rangle$ $\quad \sigma_i$ = simple loop around i^{th} pt

Note: C connected $\Longleftrightarrow$ $\operatorname{im}(\rho) \subset S_d$ transitive subgp

alg or top $\qquad\qquad$ up to inner aut

Conclude: For a subset $B \subseteq \mathbb{P}^1$ of b points,

$$\left\{ C \xrightarrow{d} \mathbb{P}^1 \text{ branched covers} \right\} / \sim \; \xleftrightarrow{\text{bij}} \left\{ \begin{array}{l} \text{gp hom } \pi_1(\mathbb{P}^1 \setminus B) \xrightarrow{\rho} S_d \\ \text{s.t. } \operatorname{im} \rho \subset S_d \text{ transitive subgp} \end{array} \right\} / \sim$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\left\{ \text{simply branched} \right\} = \left\{ \rho(\sigma_i) \in S_d \text{ transpositions} \right\}$$

$d \gg 0$

$$\begin{tikzcd} & H_{d,b} \arrow[dl] \arrow[dr, "\beta", "\text{finite étale}"'] & \supset H_{d,B} \arrow[r] & B \\ \mathcal{M}_g & & \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta & \supset B \end{tikzcd}$$

$H_{d,b} = \bigcup_B H_{d,B}$

Thm (Clebsch–Hurwitz) $H_{d,b}$ connected $/\mathbb{C}$

$\Longrightarrow \mathcal{M}_g$ conn

σ_i = simple loop around i^{th} pt

Pf • $\pi_1(\mathbb{P}^1 \setminus B) = \langle \sigma_1, \ldots, \sigma_b \mid \prod_i \sigma_i = 1 \rangle$

• $\pi_1(\mathbb{P}^1 \setminus B) \curvearrowright$ fibers of $C \to \mathbb{P}^1$ simply branched over B

• Similarly, $\pi_1(\mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta, B)$ acts on fibers of $\beta : H_{d,b} \to \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$

• $H_{d,B} := \beta^{-1}(B) = \{ C \to \mathbb{P}^1 \text{ simple branched over } B \}$

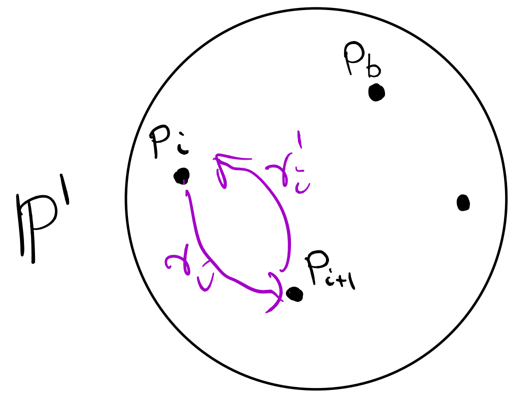
$= \left\{ \begin{array}{l} \text{gp hom } \pi_1(\mathbb{P}^1 \setminus B) \xrightarrow{\rho} S_d \text{ s.t.} \\ \mathrm{im}(\rho) \subset S_d \text{ trans } \& \ \rho(\sigma_i) \text{ transposition} \end{array} \right\}$

$= \left\{ (\tau_1, \ldots, \tau_b) \in (S_d)^b \;\middle|\; \begin{array}{l} \tau_i \text{ transposition } \& \\ \prod \tau_i = 1 \\ \langle \tau_i \rangle \subset S_d \text{ transitive} \end{array} \right\}$

$H_{d,b}$ connected $\iff \pi_1(\mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta, B) \curvearrowright H_{d,B}$ is transitive

Strategy: Find loops in $\mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$ that act on $(\tau_1, \ldots, \tau_b) \in H_{d,B}$ in a controlled way so that we can show each orbit contains

$$\tau^* = (\underbrace{(1\,2),(1\,2),(1\,3),(1\,3),\ldots,(1\ d{-}1),(1\ d{-}1)}_{2(d-1)}, \underbrace{(1\,d),(1\,d),\ldots,(1\,d)}_{2g+2})$$



Define

$\Gamma_i : [0,1] \to \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$

$t \longmapsto (p_1, \ldots, p_{i-1}, \gamma_i(t), \gamma_i'(t), p_{i+2}, \ldots)$

Check

① $\Gamma_i \cdot (\tau_1, \ldots, \tau_b) = (\tau_1, \ldots, \tau_{i-1}, \tau_i^{-1}\tau_{i+1}\tau_i, \tau_i, \tau_{i+2}, \ldots)$

② By Γ_i's in some order, can move any $(\tau_1, \ldots, \tau_b)$ to τ^*.

§3. Fulton's 1982 appendix to Harris & Morrison's admissible covers

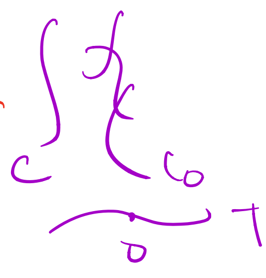
Reference: Harris & Mumford
"On the Kodaira Dimension of $\overline{M}_g$"

Completely algebraic argument in char $= 0$

Key prop Every smooth curve degenerates to a singular stable curve.

In other words, $\exists$ curve T

$$\begin{aligned} \& \; T &\longrightarrow \overline{M}_g \\ \epsilon &\longmapsto [C] \quad \text{given smooth curve} \\ 0 &\longmapsto [C_0] \quad \text{sing. stable} \end{aligned}$$



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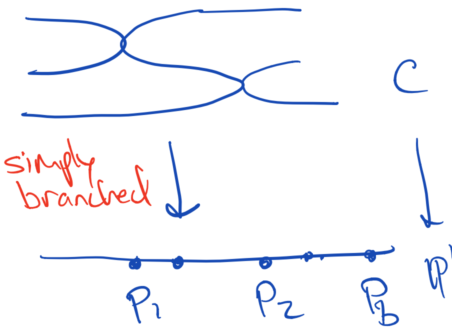
Lemma 3 In char $= 0$, $H_{d,b} \to \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$ is finite & étale, i.e. a covering space.

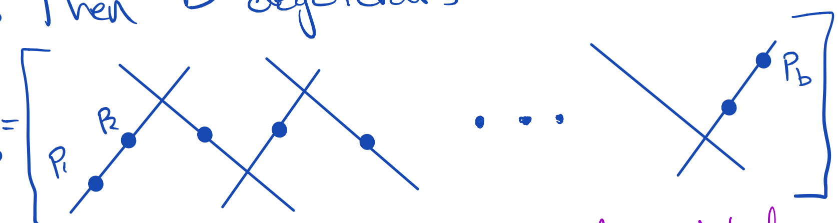
Pf of Key Prop

Take C

- Lemma 1 $\Longrightarrow$

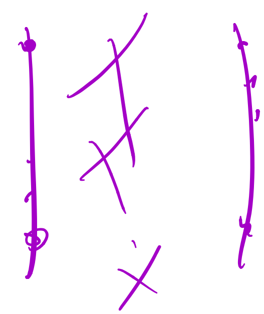


- Choose ordering $p_1, \dots, p_b \in \mathbb{P}^1$
  $\rightsquigarrow$ defines b-pointed genus 0 curve $B \in M_{0,b}$
- Lemma 3 $\Longrightarrow$ we may assume $B \in M_{0,b}$ general
- Then B degenerates to

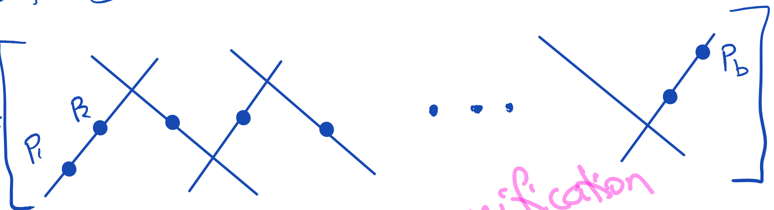
$D_0 =$  $\in \overline{M}_{0,b}$

i.e. $\exists$ family of genus 0 b-pointed stable curves

$$\begin{array}{ccccc} B & \longrightarrow & \mathcal{D} & \supset & D_0 \\ \downarrow & & \downarrow & & \\ \mathrm{Spec}\, K & \longrightarrow & \mathrm{Spec}\, R & \longleftarrow & 0 \\ & & \text{DVR} & & \end{array}$$

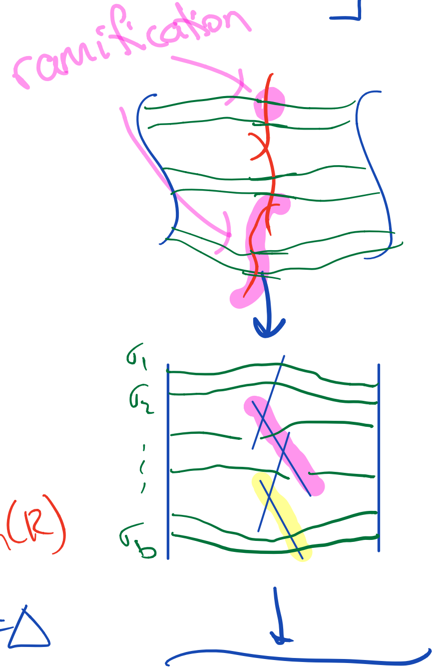


• We have constructed a degeneration of B

$D_0 = $  $\in \overline{M}_{0,b}$

• We also have

$$
\begin{tikzcd}
\mathcal{C}^\times \arrow[r, hook, "\text{smooth}"] \arrow[d, "\text{simply branched}"] & \mathcal{C} \arrow[d] \\
\mathcal{D}^\times \arrow[r, hook] \arrow[d] & \mathcal{D} \arrow[d, "\in \overline{M}_{0,n}(R)"] \\
\operatorname{Spec} K \arrow[r, hook] & \operatorname{Spec} R = \Delta
\end{tikzcd}
$$



• Define $\mathcal{C}$ as int. closure of $\mathcal{O}_{\mathcal{D}}$ in $K(\mathcal{C}^\times)$

• Purity of the branched locus

$\Longrightarrow$ the ramification of $\mathcal{C}$ over $\mathcal{D}$ is a divisor in the relative smooth locus of $\mathcal{D}$

$\Longrightarrow$ $\mathcal{C}_0 \to D_0$ ramified over $\sigma_1(0), \dots, \sigma_b(0)$ and possibly over an entire component of $\mathcal{C}_0$

• As in stable reduction, after base change by $\Delta' \to \Delta$, $t \mapsto t^m$ & replacing $\mathcal{C}$ with $\mathcal{C} \times_\Delta \Delta'$, we can arrange that $\mathcal{C}_0 \to D_0$ is ramified only over $\sigma_i(0)$ and nodes.

Check: $\mathcal{C}_0$ is nodal

$\Longrightarrow$ $\mathcal{C} \to \Delta$ family of nodal curves

• If $\mathcal{C}_0$ is stable, we win!

• Otherwise, take stable model $\mathcal{C}^{st} \to \Delta$

Check that $\mathcal{C}^{st}_0$ is not smooth

Let $T \subset \mathcal{C}_0$ irr. comp

$$
\begin{array}{ccc}
T & \subset & \mathcal{C}_0 \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \subset & D_0
\end{array}
$$

$\Longrightarrow$ every comp of T is a $\mathbb{P}^1$

$\Longrightarrow$ $\mathcal{C}_0^{st}$ is ~~nodal~~

RH $\Longrightarrow$ $2g(T) - 2 = d(-2) + R$

If $\mathbb{P}^1$ is a tail $R \le 2 + (d-1)$

bridge $\boxed{R \le 1 + 2(d-1)}$

$\Longrightarrow$ $2g(T) - 2 \le -2d + 1 + 2(d-1) = -1$

$\Longrightarrow$ $g(T) = 0$

Key prop 1 Every smooth curve degenerates to a singular stable curve. ✓

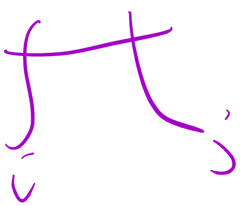
Key prop 2 $\overline{M}_g \setminus M_g$ connected

Proof $\delta := \overline{M}_g \setminus M_g = \delta_0 \cup \delta_1 \cup \cdots \cup \delta_{\lfloor g/2 \rfloor}$

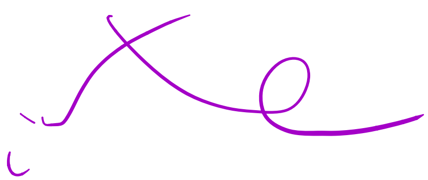
- $\delta_0 = \mathrm{im}(\overline{M}_{g-1,2} \to \overline{M}_g)$
- $\delta_i = \mathrm{im}(\overline{M}_{i,1} \times \overline{M}_{g-i,1} \to \overline{M}_g)$

By induction know each $\delta_0, \delta_1, \ldots$ is irreducible

But they intersect!



$\in \delta_i \cap \delta_j \neq \emptyset$



$\in \delta_0 \cap \delta_i \neq \emptyset$

Conclusion $\overline{M}_g$ connected

$D_0 \in M_{0,b}$

Argument shows more

limit $C_0 \to D_0$ admissible cover

$\iff$ • simply branched away from nodes

• over nodes, branching

$\mathrm{Spec}\, k[x,y]/xy \longrightarrow \mathrm{Spec}\, k[x,y]/xy$

$x \longmapsto x^m$

$y \longmapsto y^m$

stack of admissible

$$\begin{tikzcd} & \overline{\mathcal{H}}_{d,b} \arrow[dl] \arrow[dr, "\text{finite}"] & \\ \overline{M}_g & & \overline{M}_{0,b} \end{tikzcd}$$

§4. Two irreducibility papers in 1969

- Deligne & Mumford (2 arguments)
  The irreducibility of the space of curves of given genus
- Fulton
  Hurwitz schemes and Irreducibility of Moduli of Algebraic Curves

Both papers show that $\overline{\mathcal{M}}_g$ irreducible in positive characteristic ($p > g+1$ in Fulton) relying on char$=0$.

Deligne-Mumford #1

Uses $\overline{\mathcal{M}}_g \to \operatorname{Spec} \mathbb{Z}$ smooth & proper

FACT If $X \to Y$ smooth & proper, the function

$y \longmapsto$ # conn. cpts of X_y

is constant.

true if fibers are geom. normal

Since $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} \mathbb{C}$ conn

$\Longrightarrow \overline{\mathcal{M}}_g \times_{\mathbb{Z}} \mathbb{F}_p$ conn

Deligne-Mumford #2

STEP 1 For any field k of char$=p$, $\nexists$ proper conn component of $\mathcal{M} := \mathcal{M}_g \times_{\mathbb{Z}} k$

- Uses existence of cms of $\mathcal{M}_g \to M_g$ over $\mathbb{Z}$ using GIT
- Use compactification $M_g \subset X$ proj$/\mathbb{Z}$
- Using char, $X \times_{\mathbb{Z}} \mathbb{C}$ conn $\Rightarrow$ $X \times_{\mathbb{Z}} \mathbb{F}_p$ conn
- Showed M_g not proper by using degen. of Jacobians

STEP 2 $\nexists$ conn component of $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} k$ consisting entirely of smooth curves

Follows from Step 1 & stable reduction

Step 1 & 2 $\Longrightarrow$ Key prop

STEP 3 $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ connected

Fulton's 1969 argument

The Hurwitz scheme $H_{d,b}$ is defined over $\mathbb{Z}$ & there is a diagram

finite $p > g+1$

$$\begin{tikzcd}
& {[C\to\mathbb{P}^1]} \quad H_{d,b} \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr, "\text{étale}", "B"'] & & \\
C & M_g & \text{branched pts} & \mathrm{Sym}^b\mathbb{P}^1\setminus\Delta
\end{tikzcd}$$

Established a "reduction theorem":
since $H_{d,b} \longrightarrow \mathrm{Sym}^b\mathbb{P}^1\setminus\Delta$ finite étale

connectedness of $H_{d,b}\times_{\mathbb{Z}}\mathbb{C} \Longrightarrow$

connectedness of $H_{d,b}\times_{\mathbb{Z}}\overline{\mathbb{F}}_p$ for $p>g+1$