LECTURE 6: First properties of alg. spaces and stacks

Thm. *The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper and irreducible Deligne–Mumford stack of dimension $3g-3$ which admits a projective coarse moduli space.*

Where are we?

- Defined $\mathcal{M}_g$ (not $\overline{\mathcal{M}}_g$)
- $\mathcal{M}_g$ is an alg. stack

We are here!

Course outline

1. Site, sheaves & stacks
2. Alg. spaces & stacks
3. Geometry of DM stacks
4. Moduli of stable curves

§0. Review

Def Let $F \to G$ be a map of presheaves/prestacks over $Sch_{Ét}$

① We say $F \to G$ is <u>repn by schems</u> if $\forall S \to G$ from a scheme S, $F \times_G S$ is a scheme

② We say $F \to G$ is <u>repn</u> if $\forall S \to G$ from a scheme S, $F \times_G S$ is an alg. space

Picture:

$$\begin{tikzcd} F \times_G S \arrow[r] \arrow[d] & S \arrow[d] \\ F \arrow[r] & G \end{tikzcd}$$

We can discuss properties of maps repn/repn by schems.

Key defns

① An <u>alg. space</u> is a sheaf X on $Sch_{Ét}$ s.t. $\exists$ scheme U and $U \to X$ repn by schems, étale & surj

Deligne-Mumford

② A <u>DM stack</u> is a stack $\mathcal{X}$ on $Sch_{Ét}$ s.t. $\exists$ scheme U and $U \to \mathcal{X}$ representable, étale & surj

③ An <u>alg. stack</u> is a stack $\mathcal{X}$ on $Sch_{Ét}$ s.t. $\exists$ scheme U and $U \to \mathcal{X}$ representable, smooth & surj

EXAMPLES

Last time

- $[X/G]$ $\}$ algebraic
- M_g

Exer: Show the stack $\mathrm{Bun}_{r,d}(C)$ of vect. bdls on C of rank r, deg d is algebraic.

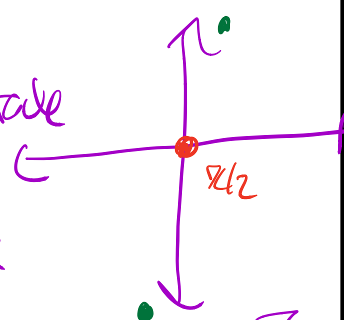
Ex 1: $G=\mathbb{Z}/2 \curvearrowright \mathbb{A}^2$, $-1\cdot(x,y)=(-x,-y)$

smooth DM stack

$$\begin{tikzcd} {[\mathbb{A}^2/G]} \arrow[d, "\substack{\text{coarse mod. space}\\ \text{birational}}"] \\ \mathbb{A}^2/G = \operatorname{Spec} k[x^2,xy,y^2] \end{tikzcd}$$

$[\mathbb{A}^2/G]$ DM stack

singular $\to$ $\mathbb{A}^2/G = \operatorname{Spec} k[x^2,xy,y^2]$ cone over a quadric in $\mathbb{P}^2$



Ex 2: $\mathbb{G}_m \curvearrowright \mathbb{A}^1$

$\mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ alg. stack, not DM

Ex 3 (Hironaka) $\exists$ smooth proper 3-fold X with a free $\mathbb{Z}/2$-action s.t. $\exists$ orbit not contained in any affine open.

$$X \xrightarrow{\text{finite}} X/(\mathbb{Z}/2)$$

alg. space (not scheme)

Ex 4: $G=\mathbb{Z}/2 \overset{\text{free}}{\curvearrowright} X$

non sep affine line.

$Y=X/G$ is alg space (not scheme)

Two reasons

① The two origins are not contained in any affine.

② The diagonal $Y \xrightarrow{\Delta} Y\times Y$ is not loc. closed imm.

$\boxed{\{(-1,x) \mid x\neq 0\} \sqcup \{1,0\}}$

$$\begin{tikzcd} G\times X \arrow[r] \arrow[d] & X\times X \arrow[d] & \mathbb{A}^1 \arrow[l] \\ Y \arrow[r] & Y\times Y & \end{tikzcd}$$

$x \mapsto (x,-x)$

Summary of important resuts (TO BE PROVEN)

Properties of the diagonal

Key pt: Diagonal encodes "stackiness"

Recall that $\forall a,b : T \to \mathcal{X}$,

$$\begin{tikzcd} \mathrm{Isom}_T(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d, "(a,b)"] \\ \mathcal{X} \arrow[r] & \mathcal{X}\times\mathcal{X} \end{tikzcd}$$

Sheaf $(S \xrightarrow{f} T) \mapsto \mathrm{Isom}_{\mathcal{X}(S)}(f^*a, f^*b)$

one of the axioms of being a stack

For $x : \mathrm{Spec}\, k \to \mathcal{X}$ for k field

define the stabilizer of x as

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & \mathrm{Spec}\, k \arrow[d, "(x,x)"] \\ \mathcal{X} \arrow[r] & \mathcal{X}\times\mathcal{X} \end{tikzcd}$$

Theorem (Representability of the diagonal)**.**

- *X alg. space $\implies X \to X \times X$ repn by schemes.*
- *$\mathcal{X}$ alg. stack $\implies \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ representable.*

In part., the stabilizer G_x is an alg. space. In fact, it is a scheme

Often, we will impose further conditions on $\mathcal{X} \to \mathcal{X}\times\mathcal{X}$.

Ex: affine, finite

Table 1: Characterization of algebraic spaces and Deligne–Mumford stacks

Type of space	Property of the diagonal	Property of stabilizers
algebraic space	monomorphism	trivial
DM stack	unramified	discrete and reduced groups FINITE
algebraic stack	arbitrary	arbitrary

Summary of important results (cont.)

Assume: Noetherian

Today

Properties of algebraic spaces

- $R \rightrightarrows X$ étale equivalence relation of schemes $\implies$ the quotient sheaf X/R is an algebraic space.
- X alg space $\implies \exists$ dense open scheme $U \subset X$.
- $X \to Y$ sep and q.fin morphism alg spaces $\implies$ $X \to Y$ quasi-affine (*Zariski's Main Theorem*).

Properties of Deligne–Mumford stacks

- $R \rightrightarrows X$ is an étale groupoid of scheme $\implies$ the quotient stack $[X/R]$ is a DM stack.
- $\mathcal{X}$ DM stack $\implies \exists$ scheme U and finite morphism $U \to \mathcal{X}$ (*Global structure of DM stacks*).
- $\mathcal{X}$ DM stack $+ x \in \mathcal{X}(k) \implies \exists$ étale ngbd of x

$$[\mathrm{Spec}(A)/G_x] \to \mathcal{X}$$

(*Local Structure of DM Stacks*).

- $\mathcal{X}$ sep DM stack $\implies \exists$ a coarse mod space $\mathcal{X} \to X$ where X is a sep algebraic space (*Keel-Mori theorem*).

§1. Properties of morphisms

Def. Let $\mathcal{P}$ be a property of maps of schemes.

- $\mathcal{P}$ is *étale-local on the source* if for any $X' \overset{\text{ét}}{\twoheadrightarrow} X$, then $X \to Y$ has $\mathcal{P}$ $\iff$ $X' \to X \to Y$ has $\mathcal{P}$.

Ex: étale, surjective

$$\begin{tikzcd} X \times_Y Y' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

- $\mathcal{P}$ is *étale-local on the target* if for any $Y' \overset{\text{ét}}{\twoheadrightarrow} Y$, then $X \to Y$ has $\mathcal{P}$ $\iff$ $X \times_Y Y' \to Y'$ has $\mathcal{P}$.

Ex: almost everything (except projectivity)

- Same defn for smooth-local on source & target

smooth-local on source $\left\{ \begin{array}{l} \text{étale} \\ \text{surj} \\ \text{smooth} \\ \text{flat} \\ \text{loc. of f. type} \end{array} \right.$

Def. Assume $\mathcal{P}$ stable under composition and base change.

(1) If $\mathcal{P}$ is étale-local on the source and target, a map $\mathcal{X} \to \mathcal{Y}$ of DM stacks *has property* $\mathcal{P}$ if for all étale presentations $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$,

$$\begin{tikzcd} U \arrow[r, two heads, "\text{ét}"] & \mathcal{X} \times_{\mathcal{Y}} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d, two heads, "\text{ét}"] \\ & \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

the composition $U \to V$ has $\mathcal{P}$.

(1') If $\mathcal{P}$ is smooth-local on source & target, can define property $\mathcal{P}$ of $\mathcal{X} \to \mathcal{Y}$ of alg. stacks

(2) A map $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks representable by schemes *has property* $\mathcal{P}$ if for every map $T \to Y$ from a scheme, the base change $\mathcal{X} \times_{\mathcal{Y}} T \to T$ has $\mathcal{P}$.

Ex: open imm, cl. imm, loc. closed imm affine, q. affine.

§2. Properties of stacks

Def. Let $\mathcal{P}$ be a property of schemes.

• $\mathcal{P}$ is *étale-local* if for any $X' \xrightarrow{\text{ét}} X$, then X has $\mathcal{P}$ $\iff$ X' has $\mathcal{P}$.

Def: We say a DM stack $\mathcal{X}$ has P $\iff$ $\forall$ étale pres $U \to \mathcal{X}$, U has P (equiv. $\exists$)

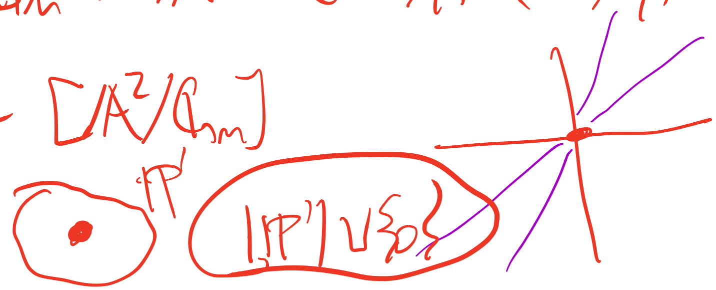
• Same for smooth-local

Ex: loc. noeth, regular, reduced are smooth-local

Upshot: Make sense of alg stack being reduced, regular, loc noeth

Ex 3 $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ $\quad t\cdot(x,y) = (tx,ty)$

$\mathbb{P}^1 \subset [\mathbb{A}^2/\mathbb{G}_m]$



Topological properties

Def (Topological space of an alg. stack $\mathcal{X}$).

$$|\mathcal{X}| := \{\operatorname{Spec} K \xrightarrow{x} \mathcal{X} \mid K \text{ is a field}\}/\sim$$

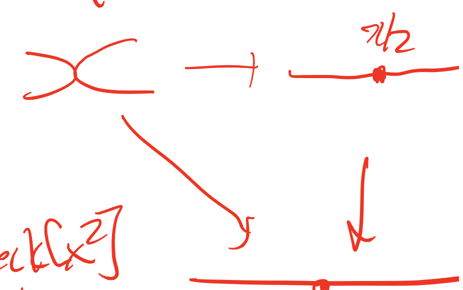
where $(\operatorname{Spec} K_1 \xrightarrow{x_1} \mathcal{X}) \sim (\operatorname{Spec} K_2 \xrightarrow{x_2} \mathcal{X})$ if $\exists\, K_1 \to K_3$ and $K_2 \to K_3$ s.t. $x_1|_{\operatorname{Spec} K_3} \xrightarrow{\sim} x_2|_{\operatorname{Spec} K_3}$.

$$\begin{tikzcd} & \operatorname{Spec} K_1 \arrow[dr, "x_1"] \arrow[dd, phantom, "\Downarrow"] & \\ \operatorname{Spec} K_3 \arrow[ur] \arrow[dr] & & \mathcal{X} \\ & \operatorname{Spec} K_2 \arrow[ur, "x_2"'] & \end{tikzcd}$$

• $U \subset |\mathcal{X}|$ is *open* if $\exists$ open imm $\mathcal{U} \hookrightarrow \mathcal{X}$ such that U is the image of $|\mathcal{U}| \to |\mathcal{X}|$.

Ex 1 $G = \mathbb{Z}/2 \curvearrowright \mathbb{A}^1$ $\quad -1\cdot x = -x$

$$\begin{tikzcd} \mathbb{A}^1 \arrow[r] \arrow[dr, "x \mapsto x^2"'] & {[\mathbb{A}^1/G]} \arrow[d] \\ & \mathbb{A}^1/G = \operatorname{Spec} k[x^2] \cong \mathbb{A}^1 \end{tikzcd}$$



$|[\mathbb{A}^1/G]| = |\mathbb{A}^1|$

Ex: $\mathbb{G}_m \curvearrowright \mathbb{A}^1$ $\quad |[\mathbb{A}^1/\mathbb{G}_m]|$

Defn: • An alg. stack $\mathcal{X}$ is quasi-compact, connected or irreducible if $|\mathcal{X}|$ is.

• A morphism $\mathcal{X} \to \mathcal{Y}$ is quasi-compact if $|\mathcal{X}| \to |\mathcal{Y}|$ is

• A morphism $\mathcal{X} \to \mathcal{Y}$ is f.type if loc. f.type & quasi-compact.

Exer: Show $\mathcal{X}$ q.compact $\iff$ $\exists$ sm. pres $\operatorname{Spec} A \twoheadrightarrow \mathcal{X}$

Exer: If $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ q.compact $\implies |\mathcal{X}|$ sober top. space

(every irred closed subset has a gen. pt)

§3. Equiv. relations & groupoids

Definition 0.0.1. An *étale groupoid of schemes* is a pair of étale maps $s, t\colon R \rightrightarrows U$ of schemes called the *source* and *target* and a *composition* morphism $c\colon R \times_{t,U,s} R \to R$ satisfying:

$R \overset{s}{\underset{t}{\rightrightarrows}} U$

(1) (associativity)

$$\begin{tikzcd} R \times_{t,U,s} R \times_{t,U,s} R \arrow[r, "c\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times c"'] & R \times_{t,U,s} R \arrow[d, "c"] \\ R \times_{t,U,s} R \arrow[r, "c"] & R, \end{tikzcd}$$

(2) (identity) $\exists\ e\colon U \to R$ such that

$$\begin{tikzcd} & U \arrow[dl, "\mathrm{id}"'] \arrow[d, "e"] \arrow[dr, "\mathrm{id}"] & \\ U & R \arrow[l, "s"] \arrow[r, "t"'] & U \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "e\circ s,\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & R \times_{t,U,s} R \arrow[d, "c"] & R \arrow[l, "e\circ t,\mathrm{id}"'] \arrow[dl, "\mathrm{id}"] \\ & R, & \end{tikzcd}$$

(3) (inverse) $\exists\ i\colon R \to R$ such that

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "s"'] & R \arrow[r, "i"] \arrow[d, "t"] & R \arrow[dl, "s"] \\ & U & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "s"] \arrow[d, "(\mathrm{id},i)"'] & U \arrow[d, "e"] \\ R \times_{t,U,s} R \arrow[r, "c"] & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "t"] \arrow[d, "(i,\mathrm{id})"'] & U \arrow[d, "e"] \\ R \times_{t,U,s} R \arrow[r, "c"] & R. \end{tikzcd}$$

If $(s, t)\colon R \to U \times U$ is a monomorphism, then we say $s, t\colon R \rightrightarrows U$ is an *étale equivalence relation.*

Same for smooth

Think of R as a "scheme of relations"

$r \in R \overset{\text{relation}}{\rightsquigarrow} s(r) \xrightarrow{r} t(r)$

Composition

$(u \xrightarrow{r} v) \circ (v \xrightarrow{r'} w) = (u \xrightarrow{r' \circ r} w)$

identity $u \xrightarrow{\mathrm{id}} u$

inverse $u \xrightarrow{r} v \rightsquigarrow v \xrightarrow{r'} u$

$R \rightrightarrows U$ equiv relation

$\Longrightarrow \exists$ at most one relation between any two points of U

Ex 1 G smooth alg. group / field k

U k-scheme w/ G-action

$$R := G \times U \underset{t = p_2}{\overset{s = \text{mult}}{\rightrightarrows}} U \quad \text{smooth}$$

(étale if G is finite)

$(g \in G \rightsquigarrow u \xrightarrow{g} g \cdot u)$

equiv relation $\iff$ $G \curvearrowright U$ free

(i.e. $G \times U \xrightarrow{\text{mono}} U \times U$)

Ex 2 Let $\mathcal{X}$ be DM stack

Let $U \to \mathcal{X}$ étale pres

$$R = U \times_{\mathcal{X}} U \underset{t = p_2}{\overset{s = p_1}{\rightrightarrows}} U$$

étale groupoid

equiv. relation $\iff$ $\mathcal{X}$ alg. space

Def Let $R \underset{t}{\overset{s}{\rightrightarrows}} U$ be a smooth groupoid

Define $[U/R]^{pre}$ as prestack

s.t. $[U/R]^{pre}(S) := [U(S)/R(S)]$

groupoid quotient

Define $[U/R]$ as stackification.

Exer: $\exists$ cart. diagram

$$\begin{tikzcd}
R \arrow[r, "s"] \arrow[d, "t"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\
U \arrow[r, "p"] & {[U/R]}
\end{tikzcd}$$

$$\begin{tikzcd}
R \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \times U \arrow[d, "p \times p"] \\
{[U/R]} \arrow[r, "\Delta"] & {[U/R] \times [U/R]}
\end{tikzcd}$$

__THM__ $R \rightrightarrows U$ étale (resp. smooth) groupoid

$\Rightarrow [U/R]$ is DM stack (resp. alg. stack)