LECTURE 7 : Representability of the diagonal

Definitions.

- An *algebraic space* is a sheaf X on $\mathrm{Sch}_{\text{Ét}}$ such that there exist a scheme U and a surjective étale morphism $U \to X$ representable by schemes.
- A *Deligne–Mumford stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{Ét}}$ such that there exist a scheme U and a surjective, étale and representable morphism $U \to \mathcal{X}$.
- An *algebraic stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{Ét}}$ such that there exist a scheme U and a surjective, smooth and representable morphism $U \to \mathcal{X}$.

§0. Review

Def. An *étale groupoid of schemes* is a pair of étale maps $s, t\colon R \rightrightarrows U$ of schemes called the *source* and *target* and a *composition* morphism $c\colon R \times_{t,U,s} R \to R$ satisfying:

(1) (associativity)

$$\begin{tikzcd}
R \times_{t,U,s} R \times_{t,U,s} R \arrow[r, "c \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times c"] & R \times_{t,U,s} R \arrow[d, "c"] \\
R \times_{t,U,s} R \arrow[r, "c"] & R,
\end{tikzcd}$$

(2) (identity) $\exists\, e\colon U \to R$ such that

$$\begin{tikzcd}
& U \arrow[dl, "\mathrm{id}"'] \arrow[d, "e"] \arrow[dr, "\mathrm{id}"] & \\
U & R \arrow[l, "s"] \arrow[r, "t"'] & U
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "e\circ s{,}\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & R \times_{t,U,s} R \arrow[d, "c"] & R \arrow[l, "e\circ t{,}\mathrm{id}"'] \arrow[dl, "\mathrm{id}"] \\
& R, &
\end{tikzcd}$$

(3) (inverse) $\exists\, i\colon R \to R$ such that

$$\begin{tikzcd}
R \arrow[r, "i"] \arrow[dr, "s"'] & R \arrow[r, "i"] \arrow[d, "t"] & R \arrow[dl, "s"] \\
& U &
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "s"] \arrow[d, "(\mathrm{id}{,}i)"] & U \arrow[d, "e"] \\
R \times_{t,U,s} R \arrow[r, "c"] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "t"] \arrow[d, "(i{,}\mathrm{id})"] & U \arrow[d, "e"] \\
R \times_{t,U,s} R \arrow[r, "c"] & R.
\end{tikzcd}$$

If $(s,t)\colon R \to U \times U$ is a monomorphism, then we say $s, t\colon R \rightrightarrows U$ is an *étale equivalence relation.*

Same defn for <u>smooth</u>

<u>Main ex:</u> $G \curvearrowright U$ scheme

$$R = G \times U \overset{\sigma}{\underset{p_2}{\rightrightarrows}} U$$

<u>Def</u> The quotient stack $[U/R]$ of a smooth groupoid $R \rightrightarrows U$ is the stackification of

$[U/R]^{pre}$ where $[U/R]^{pre}(S) = [U(S)/R(S)]$

groupoid quotient of set-theoretic groupoid $R(S) \rightrightarrows U(S)$

<u>Know</u>

$$\begin{tikzcd}
R \arrow[r, "s"] \arrow[d, "t"] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\
U \arrow[r, "p"] & {[U/R]}
\end{tikzcd}$$

cartesian

$$\begin{tikzcd}
R \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \times U \arrow[d, "p \times p"] \\
{[U/R]} \arrow[r, "\Delta"] & {[U/R] \times [U/R]}
\end{tikzcd}$$

THM $R \rightrightarrows U$ étale (resp. smooth) groupoid.

$\Longrightarrow [U/R]$ is a DM (resp. algebraic) stack

Pf Claim $U \to [U/R]$ representable

Let $T \to [U/R]$ map from a scheme

$$\begin{tikzcd}
U_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\
T \arrow[r] & {[U/R]}
\end{tikzcd}$$

Need to show:
U_T is an alg. space
& $U_T \to T$ étale (resp. smooth)

Know ∃

$$\begin{tikzcd}
T' \arrow[r] \arrow[d, two heads, "\text{ét}"'] \arrow[dr, phantom, "\Nearrow"] & U \arrow[d] \\
T \arrow[r] & {[U/R]}
\end{tikzcd}$$

Consider cube

$$\begin{tikzcd}
& \overset{\text{scheme}}{U_{T'}} \arrow[rr] \arrow[dl] \arrow[dd, "\text{q. affine}"] & & \overset{\text{scheme}}{T'} \arrow[dl] \arrow[dd, "\text{ét repn by scheme}"] \\
R \arrow[rr] \arrow[dd] & & U \arrow[dd] & \\
& \underset{\text{sheaf}}{U_T} \arrow[rr] \arrow[dl] & & \overset{\text{scheme}}{T} \arrow[dl] \\
U \arrow[rr] & & {[U/R]} &
\end{tikzcd}$$

everything cartesian except left & right

Therefore, $\underset{\text{scheme}}{U_{T'}} \to U_T$ étale & repn by schemes

$\Rightarrow U_T$ alg. space.

Rmk Similar argument

If $R \rightrightarrows U$ étale equiv. relation

& s, t are $\underline{\text{quasi-cpt \& sep}}$

$\Rightarrow U/R$ alg. space

§2. Examples

1) 3 descriptions of the \boxed{\text{"bug-eyed cover"}}

ⓐ $\mathbb{Z}/2 \curvearrowright \mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1 = U$ non-sep. affine line

$$\boxed{X = U/(\mathbb{Z}/2)}$$

ⓑ $\mathbb{Z}/2 \curvearrowright \mathbb{A}^1$ $\quad x \mapsto -x$

$R \subset \mathbb{Z}/2 \times \mathbb{A}^1$ complement of $(-1,0)$

unique nontrivial stabilizer.

$$\begin{tikzcd} R \arrow[d, shift left] \arrow[d, shift right] \\ \mathbb{A}^1 \end{tikzcd}$$

étale equiv relat

$$\boxed{X = \mathbb{A}^1/R}$$

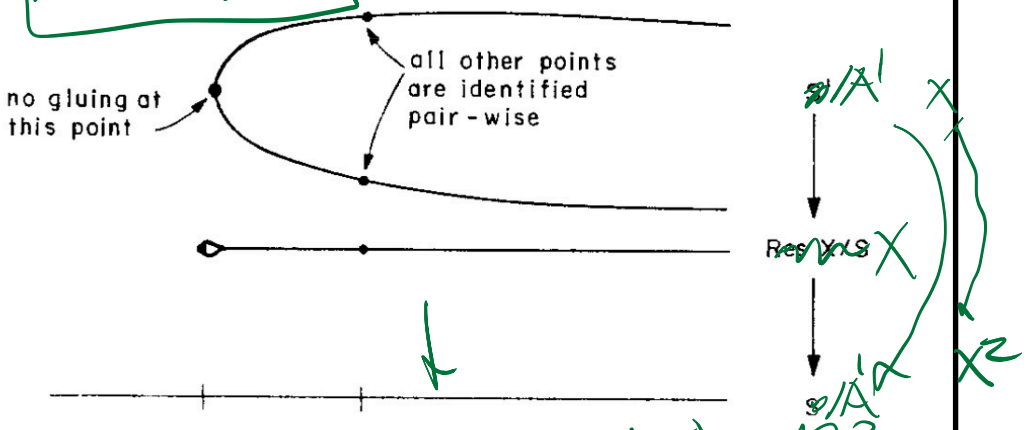
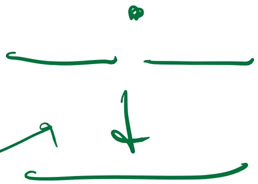


FIGURE 1.1.

Artin 172

X is not a scheme

$$\begin{tikzcd} \{(-1,z) \mid z \neq 0\} \sqcup \{(1,z)\} & \mathbb{A}^1 \arrow[d] & z \arrow[d, mapsto] \\ R \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{A}^1 \times \mathbb{A}^1 \arrow[d] & (z,-z) \\ X \arrow[r] & X \times X & \end{tikzcd}$$



not locally closed $\qquad$ not locally closed

$\implies$ X not scheme

ⓒ (Mumford)

$SL_2 \curvearrowright V_d := \mathrm{Sym}^d k^2$

$W = \left\{ (L,F) \;\middle|\; \begin{array}{l} L \neq 0 \\ F = Q^2 \text{ where} \\ Q \text{ quadric w/ disc} = 1 \end{array} \right\}$

$\subset V_1 \times V_4$

<u>Exer</u>: $X = \underset{\text{g. affine}}{W}/SL_2$

More pathological examples

② $\mathbb{Z}/2 \curvearrowright \mathbb{A}^1_{\mathbb{C}}$ via conjugation $\mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$

$R = \mathbb{Z}/2 \times \mathbb{A}^1_{\mathbb{C}} \setminus \{(-1,0)\}$ defined/$\mathbb{R}$

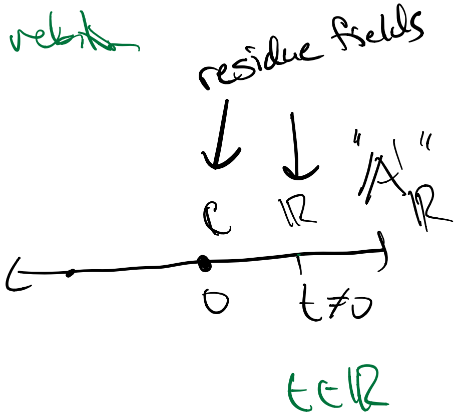
$\downdownarrows$ étale equiv relation

$\mathbb{A}^1_{\mathbb{C}}$

$\downarrow$

$X = \mathbb{A}^1_{\mathbb{C}}/R$

defined /$\mathbb{R}$



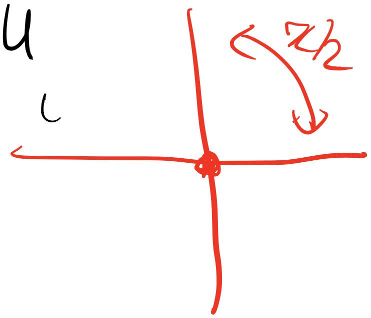
$$\begin{tikzcd}
\text{scheme } X_{\mathbb{C}} \arrow[r, "\text{ét}", "\mathbb{Z}/2\text{-tor}"'] \arrow[d] & X \text{ alg. space not scheme} \arrow[d] \\
\operatorname{Spec}\mathbb{C} \arrow[r, "\text{ét}"] & \operatorname{Spec}\mathbb{R}
\end{tikzcd}$$

Exer: $X_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{C}} \cup_{\mathbb{A}^1_{\mathbb{C}} \setminus 0} \mathbb{A}^1_{\mathbb{C}}$ ↺ $\mathbb{Z}/2$

action swaps origins & acts via conjugation on $\mathbb{A}^1_{\mathbb{C}} \setminus 0$

③ $\mathbb{Z}/2 \curvearrowright \operatorname{Spec} k[x,y]/xy = U$

via $(-1)\cdot(x,y) = (y,x)$



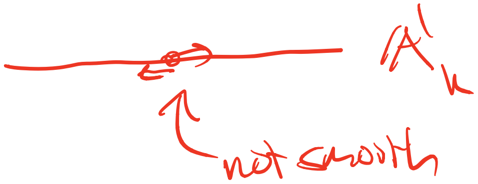
$R = \mathbb{Z}/2 \times U \setminus \{(-1,0)\}$

$\downdownarrows$ étale equiv. relation

U

$\downarrow$

$X = U/R$



④ $\operatorname{char}(k) = 0$

Consider $\mathbb{Z}$ as group scheme $/k$

$\left(\mathbb{Z} = \coprod_{n \in \mathbb{Z}} \operatorname{Spec} k\right)$

$\mathbb{Z} \curvearrowright \mathbb{A}^1_k$ via $n \cdot x = x + n$

$\mathbb{A}^1_k/\mathbb{Z}$ alg. space

not a scheme

not q.sep.

$\mathbb{Z} \times \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1$

not q.c.

Don't know: $B\mathbb{Z}$ algebraic
(but let's ignore that)

⑤ $\mathbb{Z}$ is a group scheme/k
discrete & reduced
not quasi-compact

$\mathcal{X} = B_k\mathbb{Z}$ DM stack w/ $\Delta_{\mathcal{X}/k}$ not q.compact

$\implies \mathcal{X}$ not q.sep

⑥ $G = \mathbb{A}^1_k/\mathbb{Z}$ group alg space/k

G q.compact but $\Delta_{G/k}$ is not

$\mathbb{Z} \subset \mathbb{G}_a = \mathbb{A}^1$

$\implies \mathcal{X} = BG$ q.compact, $\Delta_{\mathcal{X}}$ q.compact, loc noeth

but $\Delta_{\Delta_{\mathcal{X}}}$ not q.compact

Related ex: $G = \mathbb{A}^\infty \cup_{\mathbb{A}^\infty \setminus 0} \mathbb{A}^\infty$ scheme not q.sep

$\mathcal{X} = BG$ q.compact, $\Delta_{\mathcal{X}}$ q.compact

not loc noeth $\Delta_{\Delta_{\mathcal{X}}}$ not q.compact

⑦ $G = \mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$

not sep →

$$\begin{tikzcd} G \arrow[d, "\text{group scheme}"] \\ \mathbb{A}^1 \end{tikzcd}$$

$\mathcal{X} = B_{\mathbb{A}^1}G \to \mathbb{A}^1$

DM stack, $\mathcal{X}$, $\Delta_{\mathcal{X}}$, $\Delta_{\Delta_{\mathcal{X}}}$ q-compact

But $\Delta_{\mathcal{X}}$ not sep

(here $\Delta_{\mathcal{X}}$ is repr by schemes)

FACT: $\mathcal{X}$ DM w/ quasi-compact & sep diag

$\implies \Delta_{\mathcal{X}}$ q.affine

⑧ Example of DM stack w/ q.compact diag. but diag is not sep & not repn by schemes

$\boxed{p \neq 2}$

Need p^{th} roots

$$\begin{tikzcd} H := \mathbb{Z}/p \setminus \left\{ \begin{smallmatrix} \text{non-id} \\ \text{elements} \\ \text{over } 0 \end{smallmatrix} \right\} \arrow[rr, "\substack{\cdot\text{bij mono} \\ \cdot\text{not loc closed}}", "\cdot\text{isom}/\mathbb{Q}_p"'] \arrow[dr] & & \mu_p = G \arrow[dl] \\ & \operatorname{Spec}\mathbb{Z}_p & \\ & 0 = (p) \arrow[u] & \end{tikzcd}$$

$Q := G/H$ gp alg space

$\mathcal{X} = B_{\mathbb{Z}_p}Q$ DM stack

$\Delta_{\mathcal{X}}$ q.compact

$\Delta_{\mathcal{X}}$ not sep & not repn by schemes

§3. The diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$

Theorem (Representability of the Diagonal).

(1) *The diagonal of an algebraic space is repr. by schemes.*

(2) *The diagonal of an algebraic stack is representable.*

Pf of (1) Let X alg. space

Let $T \to X \times X$ be a map from a scheme

Consider

$$\begin{tikzcd} Q_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d] \\ X \arrow[r] & X \times X \end{tikzcd}$$

Need to show

Q_T is a scheme

Choose an étale presentation

$U \to X$ with U scheme

This is an epimorphism of sheaves

$$\begin{tikzcd} & U \times U \arrow[r] \arrow[r, phantom, "\circlearrowleft", below] & X \times X \\ \exists T' \arrow[ur] \arrow[rr, "\acute{e}t"] & & T \arrow[u] \end{tikzcd}$$

$\Rightarrow U \times U \to X \times X$ epi of sheaves

all faces cart

exc. left & right

$$\begin{tikzcd} & Q_{T'} \arrow[dl] \arrow[rr, "sep", "loc.q.finite"'] \arrow[dd] & & T' \arrow[dl] \arrow[dd, "\acute{e}t"] \\ R \arrow[rr] \arrow[dd] & & U \times U \arrow[dd] & \\ & Q_T \arrow[dl] \arrow[rr] & & T \arrow[dl] \\ X \arrow[rr] & & X \times X & \end{tikzcd}$$

scheme $Q_{T'}$; scheme T' ; scheme T ; sheaf

separated & locally quasi-finite

mono.

$$\begin{tikzcd} R \arrow[r] \arrow[dr, shift left, "\acute{e}t"'] \arrow[dr, shift right] & U \times U \arrow[d] \\ & U \end{tikzcd}$$

Consider locale square

Descent for sep & loc. q. finite

$\Rightarrow Q_T$ scheme ✓

Theorem (Representability of the Diagonal).

(1) *The diagonal of an algebraic space is repr. by schemes.*

(2) *The diagonal of an algebraic stack is representable.*

PF OF (2) Let $\mathcal{X}$ alg stack

Let $T \to \mathcal{X}\times\mathcal{X}$ be a map from a scheme

Consider

$$\begin{tikzcd} Q_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X}\times\mathcal{X} \end{tikzcd}$$

Goal: Q_T is an alg space

Choose an smooth presentation

$U \to \mathcal{X}$ with U scheme

Exer: ~~This is an epimorphism of sheaves~~

$\Rightarrow$ $U\times U \to \mathcal{X}\times\mathcal{X}$ epi of sheaves

$$\begin{tikzcd} U\times U \arrow[r] & \mathcal{X}\times\mathcal{X} \\ \exists T' \arrow[u] \arrow[r, "ét"] \arrow[ur, phantom, "\circlearrowleft"] & T \arrow[u] \end{tikzcd}$$

all faces cart exc. left & right

$$\begin{tikzcd} & Q_{T'} \arrow[rr, "\iota"] \arrow[dl] \arrow[dd, "\text{ét \& repr by schemes}"] & & T' \arrow[dl] \arrow[dd, "\text{ét, repr by scheme}"] \\ R \arrow[rr] \arrow[dd] & & U\times U \arrow[dd] & \\ & Q_T \arrow[rr] \arrow[dl] & & T \arrow[dl] \\ \mathcal{X} \arrow[rr] & & \mathcal{X}\times\mathcal{X} & \end{tikzcd}$$

scheme ($Q_{T'}$), schem (T'), sheaf (Q_T), schem (T)

$Q_{T'} \to Q_T$ étale pres.

$Q_{T'}$ scheme

$\Rightarrow Q_T$ alg. space

Cor.

(1) Any map from a scheme to an alg space is repr by schemes.

(2) Any map from a scheme to an alg stack is representable.

Pf: Consider

$$\begin{tikzcd} T \times_{\mathcal{X}} S \arrow[r] \arrow[d, "\text{scheme}"'] & S \text{ scheme} \arrow[d] \\ T \arrow[r] & \mathcal{X} \end{tikzcd}$$

Know

$$\begin{tikzcd} T \times_{\mathcal{X}} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \times T \arrow[d] \\ \mathcal{X} \arrow[r, "\text{repn}"] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

Exer. If $\mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic spaces (resp. algebraic stacks), the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by schemes (resp. representable).

Don't know: Δ sheaf + alg. stack $\Longrightarrow$ alg. space

② The diag of q. sep alg. space is quasi-affine

Cor. If $R \rightrightarrows U$ be an étale equivalence relation of schemes, then U/R is an algebraic space and $U \to U/R$ is an étale presentation.

"$U/R := [U/R]$" b/c equiv. relation

Pf: Suffices to show that the diag of $X = U/R$ is representable by schemes.

($\Longrightarrow U \to X$ repn by schems & get étale & surj by descent)

all faces cart exc. left & right

$$\begin{tikzcd} & \overset{\text{scheme}}{Q_{T'}} \arrow[rr, "\iota"] \arrow[dl] \arrow[dd] & & T' \text{ scheme} \arrow[dl] \arrow[dd, "\text{et, repn by scheme, surj}"] \\ R \arrow[rr] \arrow[dd] & & U \times U \arrow[dd] & \\ & Q_T \arrow[rr] \arrow[dl] & & T \arrow[dl] \\ X \arrow[rr] & & X \times X & \end{tikzcd}$$

sheaf

sep & loc. q. finite

Same argument

§3. Properties of the diagonal

Recall The stabilizer of $x \in \mathcal{X}(k)$ is $G_x := \underline{\mathrm{Aut}}_k(x)$

Know

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d, "{(x,x)}"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

Since Δ is representable, G_x is a group alg. space

FACT If G q.compact & q-sep group alg space$/k$, then G is a $\underline{\text{alg. group}/k}$. (f.type group scheme over k)

Exer: $\mathcal{X}$ alg. stack

$\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ loc. of f.type

The inertia stack of $\mathcal{X}$

is

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r, "\text{rep'n}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathcal{X} \arrow[r, "\text{rep'n}", "\Delta"'] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

rel. group alg space

By defn

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} k \arrow[d, "x"] \\ I_{\mathcal{X}} \arrow[r] & \mathcal{X} \end{tikzcd}$$

Exer: G finite abelian group

$G \curvearrowright$ scheme U

$$I_{[U/G]} = \coprod_{g \in G} [U^g/G]$$

$\{u \in U \mid gu = u\}$

Separation properties

Def A map $\mathcal{X} \to \mathcal{Y}$ of alg stack is quasi-separated if $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact & quasi-separated.

Def Say $\mathcal{X}$ noetherian if loc. noeth, q.compact & q.separated

Exer: G smooth affine alg grp / k

$G \curvearrowright$ scheme U / k

① If $x \in U(k)$, then stabilizer of $\operatorname{Spec} k \to [U/G]$ is the usual stabilizer G_x

② U q.sep $\Rightarrow$ $[U/G]$ q.affine

③ U has affine diag (eg sep) $\Rightarrow$ $[U/G]$ affine diag.

Ex: We showed

$$\mathcal{M}_g = [\underset{\substack{\uparrow \\ \text{q-proj}}}{H'} / PGL_n]$$

$\Rightarrow$ $\mathcal{M}_g$ has affine diag.

Later: has finite diagonal