INVARIANT RINGS THROUGH CATEGORIES

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Abstract. We formulate a notion of "geometric reductivity" in an abstract categorical setting which we refer to as adequacy. The main theorem states that the adequacy condition implies that the ring of invariants is finitely generated. This result applies to the category of modules over a bialgebra, the category of comodules over a bialgebra, and the category of quasi-coherent sheaves on an algebraic stack of finite type over an affine base.

1. Introduction

A fundamental theorem in invariant theory states that if a reductive group G over a field k acts on a finitely generated k-algebra A, then the ring of invariants A^G is finitely generated over k (see [MFK94, Appendix 1.C]). Mumford's conjecture, proven by Haboush in [Hab75], states that reductive groups are geometrically reductive; therefore this theorem is reduced to showing that the ring of invariants under an action by a geometrically reductive group is finitely generated, which was originally proved by Nagata in [Nag64].

Nagata's theorem has been generalized to various settings. In [Ses77], Seshadri showed an analogous result for an action of a "geometrically reductive" group scheme over a universally Japanese base scheme; this result was further generalized in [FvdK08]. In [BFS92], the result is generalized to an action of a "geometrically reductive" commutative Hopf algebra over a field on a coalgebra. In [KT08], an analogous result is proven for an action of a "geometrically reductive" (non-commutative) Hopf algebra over a field on an algebra. In [Alp08] and [Alp10], analogous results are shown for the invariants of certain pre-equivalence relations; moreover, [Alp10] systematically develops the theory of adequacy for algebraic stacks.

These settings share a central underlying "adequacy" property which we formulate in an abstract categorical setting. Namely, consider a homomorphism of commutative rings $R \to A$. Consider an R-linear tensor category $\mathcal{C}$ with a faithful exact R-linear tensor functor

$$F : \mathcal{C} \longrightarrow \mathrm{Mod}_A$$

such that $\mathcal{C}$ is endowed with a ring object $\mathcal{O} \in \mathrm{Ob}(\mathcal{C})$ which is a unit for the tensor product. For precise definitions, please see Situation 2.1. One can then define

$$\Gamma : \mathcal{C} \longrightarrow \mathrm{Mod}_R, \quad \mathcal{F} \longmapsto \mathrm{Mor}_{\mathcal{C}}(\mathcal{O}, \mathcal{F}).$$

Adequacy means (roughly) in this setting that Γ satisfies: if $\mathcal{A} \to \mathcal{B}$ is an epimorphism of algebras in $\mathcal{C}$ such that $F(\mathcal{A})$ and $F(\mathcal{B})$ are commutative A-algebras, and if $f \in \Gamma(\mathcal{B})$, then there exists $g \in \Gamma(\mathcal{A})$ with $g \mapsto f^n$ for some $n > 0$. The main theorem of this paper is Theorem 9.5 which states (roughly) that if Γ is adequate, then for any finite type algebra $\mathcal{A}$

(1) $\Gamma(\mathcal{A})$ is of finite type over R, and
(2) For any finite type $\mathcal{A}$-module $\mathcal{F}$, then $\Gamma(\mathcal{F})$ is a finite type $\Gamma(\mathcal{A})$-module.

Note that additional assumptions have to be imposed on the categorical setting in order to even formulate the result.

In the final sections of this paper, we show how the abstract categorical setting applies to (a) the category of modules over a bialgebra, (b) the category of comodules over a bialgebra, and (c) the category of quasi-coherent sheaves on a finite type algebraic stack over an affine base. Thus the main theorem above unifies and generalizes the results mentioned above, which was the original motivation for this research.

What is lacking in this theory is a practical criterion for adequacy. Thus we would like to ask the following questions: Is there is notion of reductivity in the categorical setting? Is there an abstract analogue of Haboush's theorem? We hope to return to these question in future research.

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Conventions. Rings are associative with 1. Abelian categories are additive categories with kernels and cokernels such that $\mathrm{Im} \cong \mathrm{Coim}$ for any morphism.

2. Setup

In this section, we introduce the types of structure we are going to work with. We keep the list of basic properties to an absolute minimum, and later we introduce additional axioms to impose.

Situation 2.1. We consider the following systems of data:

(1) $R \to A$ is a map of commutative rings,
(2) $\mathcal{C}$ is a tensor R-linear abelian category (whose bifunctor is denoted by $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and unit object in $\mathcal{C}$ is denoted by $\mathcal{O}$).
(3) $F : \mathcal{C} \to \mathrm{Mod}_A$ is a R-linear tensor functor which is faithful and exact, where Mod_A is the category of A-modules.

Remark 2.2. Let $(R \to A, \mathcal{C}, F)$ is as in Situation 2.1. Since $\mathcal{C}$ is a tensor category, for objects $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$, there are natural isomorphisms

$$\alpha_{\mathcal{F},\mathcal{G},\mathcal{H}} : (\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{H} \longrightarrow \mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{H})$$

which are unique since F is faithful. Since $F : \mathcal{C} \to \mathrm{Mod}_A$ is a tensor functor, there are natural isomorphisms

$$d_{\mathcal{F},\mathcal{G}} : F(\mathcal{F}) \otimes_A F(\mathcal{G}) \longrightarrow F(\mathcal{F} \otimes \mathcal{G}),$$

which preserve the associativity relation $\alpha_{\mathcal{F},\mathcal{G},\mathcal{H}}$ in $\mathcal{C}$ with the usual associativity of tensor products of A-modules. Moreover, there is an isomorphism $F(\mathcal{O}) \cong A$ and $\mathcal{C}$ is endowed with functorial isomorphisms $\lambda_{\mathcal{F}} : \mathcal{O} \otimes \mathcal{F} \to \mathcal{F}$ and $\rho_{\mathcal{F}} : \mathcal{F} \otimes \mathcal{O} \to \mathcal{F}$ which correspond to the usual isomorphisms $A \otimes_A M \cong M \otimes_A A \cong M$ (via the isomorphisms $F(\mathcal{O}) \cong A$, $d_{\mathcal{O},\mathcal{F}}$ and $d_{\mathcal{F},\mathcal{O}}$).

Note that in particular $\mathcal{O} \otimes \mathcal{O} = \mathcal{O}$, so that $\mathcal{O}$ is an algebra in $\mathcal{C}$ (see Section 5) and every object of $\mathcal{C}$ has the additional structure of an $\mathcal{O}$-module.

Remark 2.3. In Tannakian formalism, the functor $F : \mathcal{C} \to \mathrm{Mod}_A$ is often viewed as a *fiber functor*. If $A = R = k$ is a field and $\mathcal{C}$ is, in addition, symmetric and rigid with $\mathrm{End}(\mathcal{O}) = k$, then it is shown in [SR72] that the category $\mathcal{C}$ is equivalent (as tensor categories) to the category of linear k-representations of an affine group scheme over k (not necessarily of finite type). This result has been generalized to the non-symmetric case where then $\mathcal{C}$ is equivalent to the category of comodules of a (non-commutative) Hopf algebra; see [Ulb90].

Definition 2.4. In the situation above we define the *global sections functor* to be the functor

$$\Gamma : \mathcal{C} \longrightarrow \mathrm{Mod}_R, \quad \mathcal{F} \longmapsto \Gamma(\mathcal{F}) = \mathrm{Mor}_{\mathcal{C}}(\mathcal{O}, \mathcal{F}).$$

Note that $\Gamma(\mathcal{F}) \subset F(\mathcal{F})$ since the functor F is faithful. There are canonical maps $\Gamma(\mathcal{F}) \otimes_R \Gamma(\mathcal{G}) \longrightarrow \Gamma(\mathcal{F} \otimes \mathcal{G})$ defined by mapping the pure tensor $f \otimes g$ to the map

$$\mathcal{O} = \mathcal{O} \otimes \mathcal{O} \xrightarrow{f \otimes g} \mathcal{F} \otimes_R \mathcal{G}.$$

For any pair of objects $\mathcal{F}, \mathcal{G}$ of $\mathcal{C}$ there is a commutative diagram

$$\begin{tikzcd}
\Gamma(\mathcal{F}) \otimes_R \Gamma(\mathcal{G}) \arrow[r] \arrow[d] & \Gamma(\mathcal{F} \otimes \mathcal{G}) \arrow[d] \\
F(\mathcal{F}) \otimes_A F(\mathcal{G}) \arrow[r, equal] & F(\mathcal{F} \otimes \mathcal{G}).
\end{tikzcd}$$

In particular, there is a natural $\Gamma(\mathcal{O})$-module structure on $\Gamma(\mathcal{F})$ for every object $\mathcal{F}$ of $\mathcal{C}$.

3. Direct sums

We cannot prove much without the following axiom.

Definition 3.1. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. We introduce the following axiom:

(D) The category $\mathcal{C}$ has arbitrary direct sums, and $\otimes$, F, and Γ commute with these.

This implies that $\mathcal{C}$ has colimits and that $\otimes$, F and Γ commute with these. The condition that $\mathcal{C}$ has arbitrary colimits is often referred to as *cocomplete*.

Lemma 3.2. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1 and assume that axiom (D) holds. Then Γ has a left adjoint*

$$\mathcal{O} \otimes_R - : \mathit{Mod}_R \longrightarrow \mathcal{C}$$

with $\mathcal{O} \otimes_R R \cong \mathcal{O}$, and $F(\mathcal{O} \otimes_R M) = A \otimes_R M$. Moreover, for any object $\mathcal{F}$ of $\mathcal{C}$ there is a canonical isomorphism $\mathcal{F} \otimes (\mathcal{O} \otimes_R M) = (\mathcal{O} \otimes_R M) \otimes \mathcal{F}$ which reduces to the obvious isomorphism on applying F.

Proof. For any R-module M choose a presentation $\bigoplus_{j \in J} R \to \bigoplus_{i \in I} R \to M \to 0$ and define

$$\mathcal{O} \otimes_R M = \mathrm{Coker}(\bigoplus\nolimits_{j \in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O})$$

where the arrow is given by the same matrix as the matrix used in the presentation for M. With this definition it is clear that $F(\mathcal{O} \otimes_R M) = A \otimes_R M$. Moreover, since there is an exact sequence

$$\bigoplus\nolimits_{j \in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O} \longrightarrow \mathcal{O} \otimes_R M \longrightarrow 0$$

it is straightforward to verify that $\mathrm{Mor}_{\mathcal{C}}(\mathcal{O} \otimes_R M, \mathcal{F}) = \mathrm{Mor}_R(M, \Gamma(\mathcal{F}))$. We leave the proof of the last statement to the reader. $\square$

In the situation of the lemma we will write $M \otimes_R \mathcal{F}$ instead of the more clumsy notation $M \otimes_R \mathcal{O} \otimes \mathcal{F}$.

Remark 3.3. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1, and further assume (D) holds. By Lemma 3.2 above, we have a diagram of functors

$$\begin{tikzcd} \mathrm{Mod}_R \arrow[rr, bend left, "\mathcal{O}\otimes_R -"] \arrow[drr, "A\otimes_R -"'] & & \mathcal{C} \arrow[ll, bend left, "\Gamma"] \arrow[d, "F"] \\ & & \mathrm{Mod}_A \end{tikzcd}$$

where $F \circ (\mathcal{O} \otimes_R -) = (A \otimes_R -)$, and $\mathcal{O} \otimes_R -$ is a left adjoint to Γ.

4. Commutativity, direct products and symmetric products

Definition 4.1. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. We introduce the following axiom:

(C) $\mathcal{C}$ is a *symmetric* tensor category and $F : \mathcal{C} \to \mathrm{Mod}_A$ is a *symmetric* tensor functor.

Remark 4.2. For the notion of a symmetric category and functor, see [Mac71, Section VII.7].

Remark 4.3. Condition (C) is equivalent to requiring functorial isomorphisms $\sigma_{\mathcal{F},\mathcal{G}} : \mathcal{F} \otimes \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{F}$ such that $F(\sigma_{\mathcal{F},\mathcal{G}})$ corresponds to the usual commutativity constraint $M \otimes_A N \cong N \otimes_A M$ on A-modules via the natural isomorphisms $d_{\mathcal{F},\mathcal{G}}$ and $d_{\mathcal{G},\mathcal{F}}$ (as defined in Remark 2.2). Since F is faithful, if such maps $\sigma_{\mathcal{F},\mathcal{G}}$ exist, they are unique.

Definition 4.4. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. We introduce the following axiom:

(I) The category $\mathcal{C}$ has arbitrary direct products, and F commutes with them.

If this is the case, then the category $\mathcal{C}$ has inverse limits (i.e., $\mathcal{C}$ is complete) and the functor F commutes with them, which is why we use the letter (I) to indicate this axiom.

Definition 4.5. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. We introduce the following axiom:

(S) For every object $\mathcal{F}$ of $\mathcal{C}$ and any $n \geq 1$ there exists a quotient
$$\mathcal{F}^{\otimes n} \longrightarrow \mathrm{Sym}^n_{\mathcal{C}}(\mathcal{F})$$
such that the map of A-modules $F(\mathcal{F}^{\otimes n}) \longrightarrow F(\mathrm{Sym}^n_{\mathcal{C}}(\mathcal{F}))$ factors through the natural surjection $F(\mathcal{F})^{\otimes n} \to \mathrm{Sym}^n_A(F(\mathcal{F}))$, and such that $\mathrm{Sym}^n_{\mathcal{C}}(\mathcal{F})$ is universal with this property.

Note that if axiom (S) holds, then the universality implies the rule $\mathcal{F} \rightsquigarrow \mathrm{Sym}^n_{\mathcal{C}}(\mathcal{F})$ is a functor. Moreover, for every $n, m \geq 0$ there are canonical maps
$$\mathrm{Sym}^n_{\mathcal{C}}(\mathcal{F}) \otimes \mathrm{Sym}^m_{\mathcal{C}}(\mathcal{F}) \longrightarrow \mathrm{Sym}^{n+m}_{\mathcal{C}}(\mathcal{F}).$$

If axiom (D) holds as well, then this will turn $\bigoplus_{n\geq 0} \mathrm{Sym}^n_{\mathcal{C}}(\mathcal{F})$ into a R-weakly commutative algebra in $\mathcal{C}$ (see Definitions 5.1 and 5.6 below).

Remark 4.6. If axiom (C) holds, it is easy to see that axiom (S) holds. Indeed, if $\mathcal{F}$ is an object of $\mathcal{C}$, using the maps $\sigma_{\mathcal{F},\mathcal{F}}$ we get an action of the symmetric group S_n on n letters on $\mathcal{F}^{\otimes n}$. Thus, $\mathrm{Sym}^n_{\mathcal{C}}(\mathcal{F})$ can be defined as the cokernel of a map

$$\bigoplus\nolimits_{\tau\in S_n} \mathcal{F}^{\otimes n} \longrightarrow \mathcal{F}^{\otimes n}$$

where in the summand corresponding to τ we use the difference of the identity and the map corresponding to τ. However, in many natural settings, axiom (C) does not hold (e.g., modules over an arbitrary bialgebra; see Section 11).

In the following lemma and its proof we will use the following abuse of notation. Suppose that $\mathcal{F}$, $\mathcal{G}$ are two objects of $\mathcal{C}$, and that $\alpha : F(\mathcal{F}) \to F(\mathcal{G})$ is an A-module map. We say that α is a *morphism of* $\mathcal{C}$ if there exists a morphism $a : \mathcal{F} \to \mathcal{G}$ in $\mathcal{C}$ such that $F(a) = \alpha$. Note that if a exists it is unique.

Lemma 4.7. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1 and assume that axiom (I) holds. Let $\mathcal{F}$, $\mathcal{G}$ be two objects of $\mathcal{C}$. Let $\alpha : F(\mathcal{F}) \to F(\mathcal{G})$ be an A-module map. The functor*

$$\Psi : \mathcal{C} \longrightarrow \mathit{Sets}, \quad \mathcal{H} \longmapsto \{\varphi \in \mathit{Mor}_{\mathcal{C}}(\mathcal{G}, \mathcal{H}) \mid F(\varphi)\circ\alpha \text{ is a morphism of } \mathcal{C}\}$$

is representable by an object $\mathcal{G}'$; that is, there is a natural bijection $\mathit{Hom}(\mathcal{H}, \mathcal{G}') = \Psi(\mathcal{H})$. The universal object $\mathcal{G} \to \mathcal{G}'$ is an epimorphism.

Proof. Since $\mathcal{C}$ is abelian, any morphism $\pi : \mathcal{G} \to \mathcal{H}$ factors uniquely as $\mathcal{G} \to \mathcal{H}' \to \mathcal{H}$ where the first map π' is an epimorphism and the second is an monomorphism. If $F(\pi)\circ\alpha = F(a)$ is a morphism of $\mathcal{C}$, then a factors through $\mathcal{H}'$ and we see that $F(\pi')\circ\alpha$ is a morphism of $\mathcal{C}$. Hence it suffices to consider epimorphisms. Consider the set $T = \{\pi : \mathcal{G} \to \mathcal{H}_\pi\}$ of epimorphisms π such that $F(\pi)\circ\alpha$ is a morphism of $\mathcal{C}$. Set

$$\mathcal{G}' = \mathrm{Im}(\mathcal{G} \longrightarrow \prod\nolimits_{\pi\in T} \mathcal{H}_\pi).$$

The rest is clear. □

Lemma 4.8. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. If either axiom (C) or (I) holds, then so does (S).*

Proof. If (C) holds, then Remark 4.6 shows that axiom (S) holds. Suppose (I) holds. Let $\mathcal{F}$ be an object of $\mathcal{C}$. The quotient $\mathcal{F}^{\otimes n} \to \mathrm{Sym}^n_{\mathcal{C}}(\mathcal{F})$ is characterized by the property that if $a : \mathcal{F}^{\otimes n} \to \mathcal{G}$ is a map such that $F(a)$ factors through $F(\mathcal{F})^{\otimes n} \to \mathrm{Sym}^n_A(F(\mathcal{F}))$ then a factors in $\mathcal{C}$ through the map to $\mathrm{Sym}^n_{\mathcal{C}}(\mathcal{F})$. To prove such a quotient exists apply Lemma 4.7 to the map

$$\bigoplus\nolimits_{\tau\in S_n} F(\mathcal{F})^{\otimes n} \longrightarrow F(\mathcal{F})^{\otimes n}$$

mentioned above. □

5. Algebra objects

We recall the notation of algebras and left modules over algebras in tensor categories as defined, for instance, in [Ost03].

Definition 5.1. Let $\mathcal{C}$ be a tensor category.

(1) An *algebra in* $\mathcal{C}$ is an object $\mathcal{A}$ in $\mathcal{C}$ endowed with $\mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $e : \mathcal{O} \to \mathcal{A}$ such that the diagrams

$$\begin{tikzcd} \mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A}) \arrow[d, "\mathrm{id}\otimes\mu_{\mathcal{A}}"'] & (\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A} \arrow[l, "\alpha_{\mathcal{A},\mathcal{A},\mathcal{A}}"'] \arrow[r, "\mu_{\mathcal{A}}\otimes\mathrm{id}"] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\mu_{\mathcal{A}}"] \\ \mathcal{A} \otimes \mathcal{A} \arrow[rr, "\mu_{\mathcal{A}}"] & & \mathcal{A} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{O} \otimes \mathcal{A} \arrow[d, "e\otimes\mathrm{id}"'] \arrow[dr, "\lambda_{\mathcal{A}}"] & \\ \mathcal{A} \otimes \mathcal{A} \arrow[r, "\mu_{\mathcal{A}}"] & \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \otimes \mathcal{O} \arrow[d, "\mathrm{id}\otimes e"'] \arrow[dr, "\rho_{\mathcal{A}}"] & \\ \mathcal{A} \otimes \mathcal{A} \arrow[r, "\mu_{\mathcal{A}}"] & \mathcal{A} \end{tikzcd}$$

commute.

(2) If $\mathcal{A}$ is an algebra in $\mathcal{C}$, a *left* $\mathcal{A}$*-module* is an object $\mathcal{F}$ in $\mathcal{C}$ endowed with a morphism $\mu_{\mathcal{F}} : \mathcal{A} \otimes \mathcal{F} \to \mathcal{F}$ such that the diagrams

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{F} \arrow[r, "\mu_{\mathcal{A}}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu_{\mathcal{F}}"'] & \mathcal{A} \otimes \mathcal{F} \arrow[d, "\mu_{\mathcal{F}}"] \\ \mathcal{A} \otimes \mathcal{F} \arrow[r, "\mu_{\mathcal{F}}"] & \mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O} \otimes \mathcal{F} \arrow[d, "e\otimes\mathrm{id}"'] \arrow[dr, "\lambda_{\mathcal{F}}"] & \\ \mathcal{A} \otimes \mathcal{F} \arrow[r, "\mu_{\mathcal{F}}"] & \mathcal{F} \end{tikzcd}$$

commute.

(3) A morphism between two left $\mathcal{A}$-modules $\mathcal{F}$ and $\mathcal{G}$ is a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ in $\mathcal{C}$ such that $\varphi \circ \mu_{\mathcal{F}} = \mu_{\mathcal{G}} \circ (\mathrm{id} \otimes \varphi)$.

Remark 5.2. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. Then the structure of an algebra $\mathcal{C}$ on an object $\mathcal{A}$ is equivalent maps $\mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $e : \mathcal{O} \to \mathcal{A}$ which on applying F induce an A-algebra structure on $F(\mathcal{A})$. Similarly, a $\mathcal{A}$-module is an object $\mathcal{F}$ endowed with a morphism $\mu_{\mathcal{F}} : \mathcal{A} \otimes \mathcal{F} \to \mathcal{F}$ such that $F(\mathcal{A}) \otimes_A F(\mathcal{F}) \to F(\mathcal{F})$ induces an $F(\mathcal{A})$-module structure on $F(\mathcal{F})$, and a morphism between $\mathcal{A}$-modules $\mathcal{F}$ and $\mathcal{G}$ is a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ such that $F(\varphi) : F(\mathcal{F}) \to F(\mathcal{G})$ is a morphism of A-modules.

Let $\mathcal{A}$ be an algebra in a tensor category $\mathcal{C}$. Let $\mathrm{Mod}_{\mathcal{A}}$ denote the category of left modules over $\mathcal{A}$.

Lemma 5.3. *The category* $\mathrm{Mod}_{\mathcal{A}}$ *is abelian.*

Proof. This is well-known; see [Ost03, Lemma 3.1] or [Bru00, Lemma 3.3]. □

If $\mathcal{A}$ is an algebra in $\mathcal{C}$, then $\Gamma(\mathcal{A})$ inherits an R-algebra structure in a natural manner. In other words, we have the following diagram of rings

$$\begin{tikzcd} R \arrow[rr] \arrow[d] & & A \arrow[d] \\ \Gamma(\mathcal{O}) \arrow[r] & \Gamma(\mathcal{A}) \arrow[r] & F(\mathcal{A}) \end{tikzcd}$$

In the same vein, given a $\mathcal{A}$-module $\mathcal{F}$, the global sections $\Gamma(\mathcal{F})$ are a $\Gamma(\mathcal{A})$-module in a natural way.

Let us use $\mathrm{Hom}_{\mathcal{A}}(-,-)$ for the morphisms in the category $\mathrm{Mod}_{\mathcal{A}}$. Note that

$$\Gamma(\mathcal{F}) = \mathrm{Mor}_{\mathcal{C}}(\mathcal{O}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{F})$$

for $\mathcal{F} \in \mathrm{Mod}_{\mathcal{A}}$. The map from the left to the right associates to $f : \mathcal{O} \to \mathcal{F}$ the map

$$\mathcal{A} = \mathcal{A} \otimes \mathcal{O} \xrightarrow{1 \otimes f} \mathcal{A} \otimes \mathcal{F} \xrightarrow{\mu_{\mathcal{F}}} \mathcal{F}.$$

Lemma 5.4. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1 and assume that axiom (D) holds. Let $\mathcal{A}$ be an algebra in $\mathcal{C}$. Then the functor*

$$\Gamma : \mathit{Mod}_{\mathcal{A}} \longrightarrow \mathit{Mod}_{\Gamma(\mathcal{A})}$$

has a right adjoint

$$\mathcal{A} \otimes_{\Gamma(\mathcal{A})} - : \mathit{Mod}_{\Gamma(\mathcal{A})} \longrightarrow \mathit{Mod}_{\mathcal{A}}.$$

We have $\mathcal{A} \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{A}) = \mathcal{A}$ and $F(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} M) = F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} M$.

Proof. The proof is identical to the argument of Lemma 3.2 using that $\Gamma(\mathcal{F}) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{F})$ for any $\mathcal{A}$-module $\mathcal{F}$. $\square$

Remark 5.5. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. Assume axiom (D). Let $\mathcal{A}$ be an algebra in $\mathcal{C}$, and let S be a set. We can define the *polynomial algebra over* $\mathcal{A}$ as the algebra

$$\mathcal{A}[x_s; s \in S] = \mathcal{A} \otimes_{\Gamma(\mathcal{A})} (\Gamma(\mathcal{A})[x_s; s \in S])$$

in $\mathcal{C}$. Explicitly $\mathcal{A}[x_s; s \in S] = \bigoplus_I \mathcal{A}x^I$ where I runs over all functions $I : S \to \mathbf{Z}_{\geq 0}$ with finite support. The symbol $x^I = \prod_s x_s^{I(s)}$ indicates the corresponding monomial. The multiplication on $\mathcal{A}[x_s; s \in S]$ is defined by requiring the "elements" of $\mathcal{A}$ to commute with the variables x_s; more precisely, since axiom (C) requires that direct sums commute with tensor products, the inclusions $\mathcal{A}x^I \otimes \mathcal{A}x^J \xrightarrow{\sim} \mathcal{A}x^{I+J} \hookrightarrow \bigoplus_K \mathcal{A}x^K$ where $x^{I+J} = \prod_s x_s^{I(s)+J(s)}$ define the multiplication $(\bigoplus_I \mathcal{A}x^I) \otimes (\bigoplus_J \mathcal{A}x^J) \to \bigoplus_K \mathcal{A}x^K$.

A homomorphism $\mathcal{A}[x_s; s \in S] \to \mathcal{B}$ of algebras in $\mathcal{C}$ is given by a homomorphism $\mathcal{A} \to \mathcal{B}$ of A-algebras together with some elements $y_s \in \Gamma(\mathcal{B})$ which commute with all elements in the image of $F(\mathcal{A}) \to F(\mathcal{B})$.

Definition 5.6. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. An algebra $\mathcal{A}$ in $\mathcal{C}$ is called *F-weakly commutative* if $F(\mathcal{A})$ is commutative.

Lemma 5.7. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. If $\mathcal{A}$ is a F-weakly commutative algebra in $\mathcal{C}$ and $\mathcal{I} \subset \mathcal{A}$ is a left ideal, then $\mathcal{I}$ is a two-sided ideal and $\mathcal{A}/\mathcal{I}$ is a F-weakly commutative algebra in $\mathcal{C}$.*

Proof. Consider the image $\mathcal{I}'$ of the multiplication $\mathcal{A} \otimes \mathcal{I} \to \mathcal{A}$. By assumption $F(\mathcal{I}') = F(\mathcal{I})$, hence we have equality. The final assertion is clear. $\square$

6. Commutative algebra objects and modules

In this section, we introduce the notion of a commutative algebra in $\mathcal{C}$ where $(R \to A, \mathcal{C}, F)$ is as in Situation 2.1. This is necessary in order to define the tensor product of two modules over an algebra in $\mathcal{C}$. The goal is to construct, given an algebra $\mathcal{A}$ in $\mathcal{C}$, a tensor category $\mathrm{Mod}^c_{\mathcal{A}}$ so that $(R \to F(\mathcal{A}), \mathrm{Mod}^c_{\mathcal{A}}, F)$ is also an example of Situation 2.1. This section is not necessary for the remainder of the paper but we feel that these results are of independent interest.

Definition 6.1. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1.

(1) An algebra $\mathcal{A}$ in $\mathcal{C}$ is called *F-commutative* if there exists an isomorphism $\sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ which under F gives the usual flip isomorphism and which is compatible with the multiplication (so in particular $\mathcal{A}$ is F-weakly commutative).
(2) A module $\mathcal{F}$ over an algebra $\mathcal{A}$ in $\mathcal{C}$ is said to be *F-commutative* if there exists an isomorphism $\sigma : \mathcal{F} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{F}$ which on applying F gives the usual flip isomorphism.

It is clear that if axiom (C) holds, then any F-weakly commutative algebra $\mathcal{A}$ in $\mathcal{C}$ is F-commutative and all module over $\mathcal{A}$ are automatically F-commutative. Let us denote $\mathrm{Mod}^c_{\mathcal{A}}$ the category of all F-commutative $\mathcal{A}$-modules. This category always has cokernels, but not necessarily kernels.

Lemma 6.2. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. Let $\mathcal{A}$ be a F-commutative algebra in $\mathcal{C}$. The category $\mathrm{Mod}^c_{\mathcal{A}}$ is abelian in each of the following cases:*

(1) *axiom (C) holds, or*
(2) *the ring map $F(\mathcal{A}) \to F(\mathcal{A}) \otimes_A F(\mathcal{A})$ is flat.*

The second condition holds for example if $A \to F(\mathcal{A})$ is either flat or surjective.

Proof. In case (1) we have $\mathrm{Mod}_{\mathcal{A}} = \mathrm{Mod}^c_{\mathcal{A}}$ so the statement follows from Lemma 5.3. For case (2), let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of F-commutative $\mathcal{A}$-modules. We set $\mathcal{K} = \mathrm{Ker}(\varphi)$ and $\mathcal{Q} = \mathrm{Coker}(\varphi)$ in $\mathcal{C}$, and we know that these are kernels and cokernels in $\mathrm{Mod}_{\mathcal{A}}$. The diagram with exact rows

$$\begin{tikzcd}
\mathcal{F} \otimes \mathcal{A} \arrow[r] \arrow[d, "\sigma"] & \mathcal{G} \otimes \mathcal{A} \arrow[r] \arrow[d, "\sigma"] & \mathcal{Q} \otimes \mathcal{A} \arrow[r] \arrow[d, dotted] & 0 \\
\mathcal{A} \otimes \mathcal{F} \arrow[r] & \mathcal{A} \otimes \mathcal{G} \arrow[r] & \mathcal{A} \otimes \mathcal{Q} \arrow[r] & 0
\end{tikzcd}$$

defines the commutativity map σ for $\mathcal{Q}$. But in general we do not know that the map $\mathcal{K} \otimes \mathcal{A} \to \mathcal{F} \otimes \mathcal{A}$ is a monomorphism. After applying F this becomes the map

$$F(\mathcal{K}) \otimes_A F(\mathcal{A}) \to F(\mathcal{F}) \otimes_A F(\mathcal{A})$$

By our discussion in Section 5 we know that $B = F(\mathcal{A})$ is a F-commutative A-algebra, and $F(\mathcal{K}) \subset F(\mathcal{F})$ is an inclusion of B-modules. Note that for a B-module M we have $M \otimes_A B = M \otimes_B (B \otimes_A B)$. Hence the injectivity of the last displayed map is clear if property (2) holds, and in this case we get the commutativity restraint for $\mathcal{K}$ also. $\square$

If $\mathcal{A}$ is a F-commutative algebra in $\mathcal{C}$ and $\mathcal{F}$, $\mathcal{G}$ are $\mathcal{A}$-modules, and $\mathcal{F}$ is F-commutative then we define

$$\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G} := \begin{array}{c}\text{Coequalizer of}\\ \text{going around}\\ \text{both ways}\end{array} \left(\begin{tikzcd} \mathcal{A} \otimes \mathcal{F} \otimes \mathcal{G} \arrow[r, "\sigma\otimes 1"] \arrow[d, "\mu\otimes 1"] & \mathcal{F} \otimes \mathcal{A} \otimes \mathcal{G} \arrow[d, "1\otimes\mu"] \\ \mathcal{F} \otimes \mathcal{G} \arrow[r, "1"] & \mathcal{F} \otimes \mathcal{G} \end{tikzcd} \right)$$

Then it is clear that there is a canonical isomorphism

$$\gamma_{\mathcal{A}} : F(\mathcal{F}) \otimes_{F(\mathcal{A})} F(\mathcal{G}) \longrightarrow F(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G})$$

which is functorial in the pair $(\mathcal{F}, \mathcal{G})$. In particular, it is clear that there are functorial isomorphisms

$$\mu_{\mathcal{A}} : \mathcal{A} \otimes_{\mathcal{A}} \mathcal{F} \longrightarrow \mathcal{F}, \quad \mu_{\mathcal{A}} : \mathcal{F} \otimes_{\mathcal{A}} \mathcal{A} \longrightarrow \mathcal{F}$$

for any F-commutative $\mathcal{A}$-module $\mathcal{F}$ (via σ and the multiplication map for $\mathcal{F}$).

Lemma 6.3. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. Let $\mathcal{A}$ be an algebra in $\mathcal{C}$. Assume the category $Mod^c_{\mathcal{A}}$ is abelian. Then*

$$(R \to F(\mathcal{A}), Mod^c_{\mathcal{A}}, \otimes_{\mathcal{A}}, F, \gamma_{\mathcal{A}}, \mathcal{A}, \mu_{\mathcal{A}})$$

is another set of data as in Situation 2.1. Furthermore, if axiom (D) is satisfied for $(R \to A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$, then it is also satisfied for $(R \to F(\mathcal{A}), Mod^c_{\mathcal{A}}, \otimes_{\mathcal{A}}, F, \gamma_{\mathcal{A}}, \mathcal{A}, \mu_{\mathcal{A}})$.

Proof. This is clear from the discussion above. □

In the situation of the lemma we have the global sections functor

$$\Gamma_{\mathcal{A}} : \mathrm{Mod}_{\mathcal{A}} \longrightarrow \mathrm{Mod}_R, \quad \mathcal{F} \longmapsto \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{F}).$$

We have seen in Section 5 that for an object $\mathcal{F} \in \mathrm{Mod}_{\mathcal{A}}$ we have $\Gamma_{\mathcal{A}}(\mathcal{F}) = \Gamma(\mathcal{F})$ as R-modules.

7. Adequacy

The notion of *adequacy* is our analogue of geometric reductivity. We show in this section that it can be formulated in a variety of different ways.

Definition 7.1. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. We introduce the following axiom:

(N) The ring A is Noetherian.

Definition 7.2. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1.

(1) An object $\mathcal{F}$ of $\mathcal{C}$ is said to be *of finite type* if $F(\mathcal{F})$ is a finitely generated A-module.
(2) An algebra $\mathcal{A}$ in $\mathcal{C}$ is said to be *of finite type* if $F(\mathcal{A})$ is a finitely generated A-algebra.
(3) If $\mathcal{A}$ is an algebra in $\mathcal{C}$, an $\mathcal{A}$-module $\mathcal{F}$ is said to be *of finite type* if $F(\mathcal{F})$ is of finite type over $F(\mathcal{A})$.

Note that the algebras $\mathcal{A}$ and $F(\mathcal{A})$ in this definition need not be commutative. A noncommutative algebra S over A is finitely generated if it is isomorphic to a quotient of the free algebra $A\langle x_1, \ldots, x_n\rangle$ for some n.

Definition 7.3. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. An object $\mathcal{F}$ of $\mathcal{C}$ is called *locally finite* if it is a filtered colimit $\mathcal{F} = \text{colim } \mathcal{F}_i$ of finite type objects $\mathcal{F}_i$ such that also $F(\mathcal{F}) = \text{colim } F(\mathcal{F}_i)$.

Definition 7.4. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. We introduce the axiom:

(L) Every object $\mathcal{F}$ of $\mathcal{C}$ is locally finite.

Lemma 7.5. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. A quotient of a locally finite object of $\mathcal{C}$ is locally finite. If axioms (N) and (D) hold, then a subobject of a locally finite object is locally finite and the subcategory of locally finite objects is abelian.*

Proof. Suppose that $\mathcal{F} \to \mathcal{Q}$ is an epimorphism and that $\mathcal{F}$ is locally finite. Write $\mathcal{F} = \text{colim } \mathcal{F}_i$ of finite type objects $\mathcal{F}_i$ such that also $F(\mathcal{F}) = \text{colim } F(\mathcal{F}_i)$. Set $\mathcal{Q}_i = \text{Im}(\mathcal{F}_i \to \mathcal{Q})$. We claim that $\mathcal{Q} = \text{colim}_i\ \mathcal{Q}_i$ and that $F(\mathcal{Q}) = \text{colim } F(\mathcal{Q}_i)$. The last statement follows from exactness of F and the fact that colimits commute with images in Mod_A. If $\beta_i : \mathcal{Q}_i \to \mathcal{G}$ is a compatible system of maps to an object of $\mathcal{C}$, then composing with the epimorphisms $\mathcal{F}_i \to \mathcal{Q}_i$ gives a compatible system of maps also, whence a morphism $\beta : \mathcal{F} \to \mathcal{G}$. But $F(\beta)$ factors through $F(\mathcal{F}) \to F(\mathcal{Q})$ and hence is zero on $F(\text{Ker}(\mathcal{F} \to \mathcal{Q}))$. Because F is faithful and exact we see that β factors as $\mathcal{Q} \to \mathcal{G}$ as desired.

Suppose that $\mathcal{J} \to \mathcal{F}$ is a monomorphism, that $\mathcal{F}$ is locally finite and that (N) and (D) hold. Write $\mathcal{F} = \text{colim } \mathcal{F}_i$ of finite type objects $\mathcal{F}_i$ such that also $F(\mathcal{F}) = \text{colim } F(\mathcal{F}_i)$. Set $\mathcal{J}_i = \mathcal{F}_i \cap \mathcal{J}$. Since axiom (N) holds we see that each $\mathcal{J}_i$ is of finite type. As F is exact we see that $\text{colim } F(\mathcal{J}_i) = F(\mathcal{J})$. As axiom (D) holds we know that $\mathcal{J}' = \text{colim}\mathcal{J}_i$ exists and $\text{colim } F(\mathcal{J}_i) = F(\mathcal{J}')$. Hence we get a canonical map $\mathcal{J}' \to \mathcal{J}$ which has to be an isomorphism as F is exact and faithful. This proves that $\mathcal{J}$ is locally finite.

Assume (N) and (D). Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of locally finite objects. We have to show that the kernel and cokernel of α are locally finite. This is clear by the results of the preceding two paragraphs. $\square$

Lemma 7.6. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1 and assume that axiom (D) holds. The tensor product of locally finite objects is locally finite. For any R-module M the object $M \otimes_R \mathcal{O}$ is locally finite. If $\mathcal{A}$ is a locally finite algebra in $\mathcal{C}$, then $\mathcal{A} \otimes_{\Gamma(\mathcal{A})} M$ is locally finite for any $\Gamma(\mathcal{A})$-module M.*

Proof. This is clear since in the presence of (D), the tensor product commutes with colimits. $\square$

Recall from Definition 5.6 that an algebra $\mathcal{A}$ in $\mathcal{C}$ (as in Situation 2.1) is called *F-weakly commutative* if $F(\mathcal{A})$ is commutative.

Lemma 7.7. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1 and assume that axiom (S) holds. Consider the following conditions*

1. *For every epimorphism of finite type objects $\mathcal{G} \to \mathcal{F}$ and $f \in \Gamma(\mathcal{F})$ there exists an $n > 0$ and a $g \in \Gamma(Sym^n_{\mathcal{C}}(\mathcal{G}))$ such that $g \mapsto f^n$ in $\Gamma(Sym^n_{\mathcal{C}}(\mathcal{F}))$.*
2. *For every epimorphism $\mathcal{G} \to \mathcal{O}$ with $\mathcal{G}$ of finite type and $f \in \Gamma(\mathcal{O})$ there exists an $n > 0$ and a $g \in \Gamma(Sym^n_{\mathcal{C}}(\mathcal{G}))$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{O})$.*
3. *For every epimorphism $\mathcal{A} \to \mathcal{B}$ of F-weakly commutative algebras in $\mathcal{C}$ with $\mathcal{A}$ locally finite, and any $f \in \Gamma(\mathcal{B})$, there exists an $n > 0$ and an element $g \in \Gamma(\mathcal{A})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{B})$.*

We always have (1) $\Rightarrow$ (2) *and* (1) $\Rightarrow$ (3). *If axiom (N) holds, then* (2) $\Rightarrow$ (1). *If axiom (D) holds, then* (3) $\Rightarrow$ (1). *Furthermore, consider the following variations*

(1') *For every epimorphism of objects* $\mathcal{G} \to \mathcal{F}$ *and* $f \in \Gamma(\mathcal{F})$ *there exists an* $n > 0$ *and a* $g \in \Gamma(Sym^n_{\mathcal{C}}(\mathcal{G}))$ *which maps to* f^n *in* $\Gamma(Sym^n_{\mathcal{C}}(\mathcal{F}))$.

(2') *For every epimorphism* $\mathcal{G} \to \mathcal{O}$ *and* $f \in \Gamma(\mathcal{O})$ *there exists an* $n > 0$ *and a* $g \in \Gamma(Sym^n_{\mathcal{C}}(\mathcal{G}))$ *which maps to* f^n *in* $\Gamma(\mathcal{O})$.

(3') *For every epimorphism* $\mathcal{A} \to \mathcal{B}$ *of* F*-weakly commutative algebras in* $\mathcal{C}$, *and any* $f \in \Gamma(\mathcal{B})$, *there exists an* $n > 0$ *and an element* $g \in \Gamma(\mathcal{A})$ *such that* $g \mapsto f^n$ *in* $\Gamma(\mathcal{B})$.

If axiom (L) holds, then $(1) \Leftrightarrow (1')$, $(2) \Leftrightarrow (2')$, *and* $(3) \Leftrightarrow (3')$.

Proof. It is clear that (1) implies (2). Assume (N) + (2) and let us prove (1). Consider $\mathcal{G} \to \mathcal{F}$ and f as in (1). Let $\mathcal{H} = \mathcal{G} \times_{\mathcal{F}} \mathcal{O}$. Then $\mathcal{H} \to \mathcal{O}$ is an epimorphism, and $F(\mathcal{H}) = F(\mathcal{G}) \times_{F(\mathcal{F})} A$. By assumption (N) this implies that $F(\mathcal{H})$ is a finite A-module.

Let us prove that (1) implies (3). Let $\mathcal{A} \to \mathcal{B}$ and f be as in (3). Write $\mathcal{A} = \mathrm{colim}_i\, \mathcal{G}_i$ as a directed colimit such that $F(\mathcal{A}) = \mathrm{colim}_i\, F(\mathcal{G}_i)$ and such that each $\mathcal{G}_i$ is of finite type. Think of $f \in \Gamma(\mathcal{B}) \subset F(\mathcal{B})$. Then for some i there exists a $\tilde{f} \in F(\mathcal{G}_i)$ which maps to f. Set $\mathcal{G} = \mathcal{G}_i$, set $\mathcal{F} = \mathrm{Im}(\mathcal{G}_i \to \mathcal{B})$. The map $\mathcal{G} \to \mathcal{F}$ is an epimorphism. Since F is exact we see that $f \in F(\mathcal{F}) \subset F(\mathcal{B})$. Hence, as Γ is left exact we conclude that $f \in \Gamma(\mathcal{F})$ as well. Thus property (1) applies and we find an $n > 0$ and a $g \in \Gamma(\mathrm{Sym}^n_{\mathcal{C}}(\mathcal{G}))$ which maps to f^n in $\Gamma(\mathrm{Sym}^n_{\mathcal{C}}(\mathcal{F}))$. Since $\mathcal{A}$ and $\mathcal{B}$ are algebras in $\mathcal{C}$ we obtain a canonical diagram

$$\begin{array}{ccc} \mathcal{G}^{\otimes n} & \longrightarrow & \mathcal{F}^{\otimes n} \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{B} \end{array}$$

Since $\mathcal{A}$ and $\mathcal{B}$ are F-weakly commutative this produces a commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^n_{\mathcal{C}}(\mathcal{G}) & \longrightarrow & \mathrm{Sym}^n_{\mathcal{C}}(\mathcal{F}) \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{B} \end{array}$$

Hence the element $g \in \Gamma(\mathrm{Sym}^n_{\mathcal{C}}(\mathcal{G}))$ maps to the desired element of $\Gamma(\mathcal{A})$.

If (D) holds, then given $\mathcal{G} \to \mathcal{F}$ as in (1) we can form the map of "symmetric" algebras
$$\mathrm{Sym}^*_{\mathcal{C}}(\mathcal{G}) \longrightarrow \mathrm{Sym}^*_{\mathcal{C}}(\mathcal{F})$$
and we see that (3) implies (1).

The final statement is clear. $\square$

There are natural examples $(R \to A, \mathcal{C}, F)$ as in Situation 2.1 where axiom (S) does not hold such as the category of comodules over a general bialgebra. Hence we take property (3) of the lemma above as the defining property, since it also make sense in those situations.

Definition 7.8. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. We introduce the following axiom (A) which is the *adequacy* condition:

(A) For every epimorphism of F-weakly commutative rings $\mathcal{A} \to \mathcal{B}$ in $\mathcal{C}$ with $\mathcal{A}$ locally finite, and any $f \in \Gamma(\mathcal{B})$, there exists an $n > 0$ and an element $g \in \Gamma(\mathcal{A})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{B})$.

8. Preliminary results

Let $\mathcal{A}$ be a F-weakly commutative algebra in $\mathcal{C}$. This implies that $\Gamma(\mathcal{A}) \subset F(\mathcal{A})$ is a commutative ring. Let $I \subset \Gamma(\mathcal{A})$ be an ideal. Assuming the axiom (D) we have the object $\mathcal{A} \otimes_{\Gamma(\mathcal{A})} I$ (see Lemma 5.4) and a canonical map

$$\mathcal{A} \otimes_{\Gamma(\mathcal{A})} I \longrightarrow \mathcal{A}. \tag{8.1}$$

Namely, this is the adjoint to the map $I \to \Gamma(\mathcal{A})$. Applying F to the the map (8.1) gives the obvious map $F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} I \to F(\mathcal{A})$. The image of (8.1) will be denoted $\mathcal{A}I$ in the sequel. We have $F(\mathcal{A}I) = F(\mathcal{A})I$ by exactness of the functor F.

For an ideal I of a commutative ring B we set

$$I^* = \{f \in B \mid \exists n > 0,\ f^n \in I^n\}.$$

Note that it is not clear (or even true) in general that I^* is an ideal. (Our notation is not compatible with notation concerning integral closure of ideals in algebra texts. We will only use this notation in this section.)

Lemma 8.1. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1 and assume that axiom (D) holds. Let $\mathcal{A}$ be a locally finite, F-weakly commutative ring object of $\mathcal{C}$. Let $I \subset \Gamma(\mathcal{A})$ be an ideal. Consider the ring map*

$$\varphi : \Gamma(\mathcal{A})/I \longrightarrow \Gamma(\mathcal{A}/\mathcal{A}I).$$

If the axiom (A) holds, then

1. *the kernel of φ is contained in $I^*\Gamma(\mathcal{A})/I$; in particular it is locally nilpotent, and*
2. *for every element $f \in \Gamma(\mathcal{A}/\mathcal{A}I)$ there exists an integer $n > 0$ and an element $g \in \Gamma(\mathcal{A})/I$ which maps to f^n via φ.*

Proof. The algebra $\mathcal{A}/\mathcal{A}I$ is F-weakly commutative (by Lemma 5.7). Hence (2) is implied by axiom (A).

Suppose that $f \in \Gamma(\mathcal{A})$ maps to zero in $\Gamma(\mathcal{A}/\mathcal{A}I)$. This means that $f \in \Gamma(\mathcal{A}I)$. Choose generators $f_s \in I$, $s \in S$ for I. Consider the ring map

$$\mathcal{A}[x_s; s \in S] \longrightarrow \mathcal{B} = \bigoplus I^n\mathcal{A}$$

which maps x_s to $f_s \in \Gamma(I\mathcal{A})$, see Remark 5.5. This is an epimorphism of algebras in $\mathcal{C}$. Clearly the polynomial algebra $\mathcal{A}[x_s; s \in S]$ is F-weakly commutative and locally finite. Hence (A) implies there exists an $n > 0$ and an element

$$g \in \Gamma(\mathcal{A}[x_s; s \in S])$$

which maps to f^n in the summand $\Gamma(\mathcal{A}I^n)$ of $\Gamma(\mathcal{B})$. Hence we may also assume that g is in the degree n summand

$$\Gamma(\bigoplus\nolimits_{|J|=n} \mathcal{A}x^J)$$

of $\Gamma(\mathcal{A}[x_s; s \in S])$. Now, note that there is a ring map $\mathcal{B} \to \mathcal{A}$ and that the composition

$$\mathcal{A}[x_s; s \in S] \longrightarrow \mathcal{B} \longrightarrow \mathcal{A}$$

in degree n maps $\Gamma(\bigoplus_{|J|=n} \mathcal{A}x^J)$ into $\Gamma(\mathcal{A})I^n$, because x_s maps to f_s. Hence $f^n \in I^n$. This finishes the proof. $\square$

Let $\mathcal{A}$ be a F-weakly commutative algebra in $\mathcal{C}$. Let $\Gamma(\mathcal{A}) \to \Gamma'$ be a homomorphism of commutative rings. Write $\Gamma' = \Gamma(\mathcal{A})[x_s; s \in S]/I$. Assume axiom (D) holds. Then we see that we have the equality

$$\mathcal{A} \otimes_{\Gamma(\mathcal{A})} \Gamma' = \mathcal{A}[x_s; s \in S]/(\mathcal{A}[x_s; s \in S])I$$

where the polynomial algebra is as in Remark 5.5 and the tensor product as in Lemma 5.4. The reason is that there is an obvious map (from right to left) and that we have

$$F(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} \Gamma') = F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} \Gamma' = F(\mathcal{A})[x_s; s \in S]/(F(\mathcal{A})[x_s; s \in S])I$$

by the properties of the functor F and the results mentioned above. Hence $\mathcal{A} \otimes_{\Gamma(\mathcal{A})} \Gamma'$ is a F-weakly commutative algebra in $\mathcal{C}$ (see Lemma 5.7). Note that if $\mathcal{A}$ is locally finite, then so is $\mathcal{A} \otimes_{\Gamma(\mathcal{A})} \Gamma'$, see Lemma 7.6.

Lemma 8.2. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1 and assume that axiom (D) holds. Let $\mathcal{A}$ be an algebra in $\mathcal{C}$. Assume the axiom (A) holds, and that $\mathcal{A}$ is locally finite and F-weakly commutative. Let $\Gamma(\mathcal{A}) \to \Gamma'$ be a commutative ring map. Consider the adjunction map*

$$\varphi : \Gamma' \longrightarrow \Gamma(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} \Gamma')$$

1. *the kernel of φ is locally nilpotent, and*
2. *for every element $f \in \Gamma(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} \Gamma')$ there exists an integer $n > 0$ and an element $g \in \Gamma'$ which maps to f^n via φ.*

Proof. The homomorphism φ is an isomorphism when Γ' is a polynomial algebra (since we are assuming all functors commute with direct sums). And the general case follows from this, the discussion above the lemma and Lemma 8.1. $\square$

Lemma 8.3. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1 and assume that axioms (D) and (A) hold. Then for every locally finite, F-weakly commutative algebra $\mathcal{A}$ of $\mathcal{C}$ the map*

$$\mathit{Spec}(F(\mathcal{A})) \longrightarrow \mathit{Spec}(\Gamma(\mathcal{A}))$$

is surjective.

Proof. Let $\Gamma(\mathcal{A}) \to K$ be a ring map to a field. We have to show that the ring

$$F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} K = F(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} K)$$

is not zero. This follows from Lemma 8.2 and the fact that K is not the zero ring. $\square$

Lemma 8.4. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. Let $\mathcal{A}$ be an algebra in $\mathcal{C}$ and assume that*

1. *axioms (D) and (A) hold,*
2. *$\mathcal{A}$ is locally finite and F-weakly commutative,*
3. *$R \to A$ is finite type,*
4. *$\mathcal{A}$ is of finite type, and*
5. *$\Gamma(\mathcal{A})$ is Noetherian.*

Then $\mathit{Spec}(F(\mathcal{A})) \to \mathit{Spec}(\Gamma(\mathcal{A}))$ *is universally submersive; that is, for every discrete valuation ring* V *and every morphism* $\mathit{Spec}(V) \to \mathit{Spec}(\Gamma(\mathcal{A}))$ *there exists a local map of discrete valuation rings* $V \to V'$ *and a morphism* $\mathit{Spec}(V') \to \mathit{Spec}(F(\mathcal{A}))$ *such that*

$$\begin{tikzcd} \mathit{Spec}(F(\mathcal{A})) \arrow[d] & \mathit{Spec}(V') \arrow[l] \arrow[d] \\ \mathit{Spec}(\Gamma(\mathcal{A})) & \mathit{Spec}(V) \arrow[l] \end{tikzcd}$$

is commutative.

Proof. Let $\mathrm{Spec}(V) \to \mathrm{Spec}(\Gamma(\mathcal{A}))$ be a morphism where V is a valuation ring with fraction field K. We must show that

$$f : \mathrm{Spec}(F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} V) \longrightarrow \mathrm{Spec}(V)$$

is universally submersive. Let $\eta \in \mathrm{Spec}(V)$ be the generic point. It suffices to show that the closure of $f^{-1}(\eta)$ in $\mathrm{Spec}(F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} V)$ surjects onto $\mathrm{Spec}(V)$. If we set

$$\mathcal{I} = \ker(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} V \longrightarrow \mathcal{A} \otimes_{\Gamma(\mathcal{A})} K)$$

then $F(\mathcal{I})$ is the kernel of $F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} V \to F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} K$ and defines the closure of $f^{-1}(\eta)$. The algebra $(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} V)/\mathcal{I}$ is F-weakly commutative and locally finite. By Lemma 8.3,

$$\mathrm{Spec}(F((\mathcal{A} \otimes_{\Gamma(\mathcal{A})} V)/\mathcal{I})) \longrightarrow \mathrm{Spec}(\Gamma((\mathcal{A} \otimes_{\Gamma(\mathcal{A})} V)/\mathcal{I}))$$

is surjective. Axiom (A) applied to the epimorphism $\mathcal{A} \otimes_{\Gamma(\mathcal{A})} V \to (\mathcal{A} \otimes_{\Gamma(\mathcal{A})} V)/\mathcal{I}$ implies that

$$\mathrm{Spec}(\Gamma((\mathcal{A} \otimes_{\Gamma(\mathcal{A})} V)/\mathcal{I})) \longrightarrow \mathrm{Spec}(V)$$

is integral. Therefore the composition of the two morphisms above is surjective so that the closure of $f^{-1}(\eta)$ surjects onto $\mathrm{Spec}(V)$. □

Below we will use the following algebraic result to get finite generation.

Theorem 8.5. *Consider ring maps* $R \to B \to A$ *such that*

(1) B *and* R *are noetherian,*
(2) $R \to A$ *is of finite type, and*
(3) $\mathit{Spec}(A) \to \mathit{Spec}(B)$ *is universally submersive.*

Then $R \to B$ *is of finite type.*

Proof. The morphism $X = \mathrm{Spec}(A) \to \mathrm{Spec}(B) = Y$ is flat over a nonempty open subscheme $U \subset \mathrm{Spec}(B)$. By [RG71, Theorem 5.2.2], there exists a U-admissible blowup

$$b : \tilde{Y} \longrightarrow Y = \mathrm{Spec}(B)$$

such that the strict transform X' of X is flat over $\tilde{Y}$. For every point $y \in \tilde{Y}$ we can find a discrete valuation ring V and morphism $\mathrm{Spec}(V) \to \tilde{Y}$ whose generic point maps into U and whose special point maps to y. By assumption there exists a local map of discrete valuation rings $V \to V'$ and a commutative diagram

$$\begin{tikzcd} X \arrow[d] & & \mathrm{Spec}(V') \arrow[ll] \arrow[d] \\ Y & \tilde{Y} \arrow[l] & \mathrm{Spec}(V) \arrow[l] \end{tikzcd}$$

By definition of the strict transform we see that the product map $\text{Spec}(V') \to \tilde{Y} \times_Y X$ maps into the strict transform. Hence we conclude there exists a point on X' which maps to y, i.e., we see that $X' \to \tilde{Y}$ is surjective. By [Gro67, IV.2.7.1], we conclude that $\tilde{Y} \to \text{Spec}(R)$ is of finite type.

Let $I \subset B$ be an ideal such that $\tilde{Y}$ is the blowup of $\text{Spec}(B)$ in I. Choose generators $f_i \in I$, $i = 1, \ldots, n$. For each I the affine ring

$$B_i = B[f_j/f_i;\ j = 1, \ldots, \hat{i}, \ldots n] \subset f.f.(B)$$

in the blowup is of finite type over R. Write $B = \text{colim}_{\lambda \in \Lambda} B_\lambda$ as the union of its finitely generated R-subalgebras. After shrinking Λ we may assume that each B_λ contains f_i for all i. Set $I_\lambda = \sum f_i B_\lambda \subset B_\lambda$ and let

$$B_{\lambda,i} = B_\lambda[f_j/f_i;\ j = 1, \ldots, \hat{i}, \ldots n] \subset f.f.(B_\lambda) \subset f.f.(B)$$

After shrinking Λ we may assume that the canonical maps $B_{\lambda,i} \to B_i$ are surjective for each i (as B_i is finitely generated over R). Hence for such a λ we have $B_{\lambda,i} = B_i$! So for such a λ the blowup of $\text{Spec}(B_\lambda)$ in I_λ is **equal** to the blowup of $\text{Spec}(B)$ in I. Set $Y_\lambda = \text{Spec}(B_\lambda)$ Thus the composition

$$\tilde{Y} \longrightarrow Y \longrightarrow Y_\lambda$$

is a projective morphism and we see that

$$(Y \to Y_\lambda)_* \mathcal{O}_Y \subset (\tilde{Y} \to Y_\lambda)_* \mathcal{O}_{\tilde{Y}}$$

and the last sheaf is a coherent $\mathcal{O}_{Y_\lambda}$-module [Gro67, III.3.2.1]. Hence $(Y \to Y_\lambda)_* \mathcal{O}_Y$ is also coherent so that $Y \to Y_\lambda$ is finite which finishes the proof. □

9. The main result

The main argument in the proof of Theorem 9.5 is an induction argument. In order to formulate it, we use the following condition.

Definition 9.1. Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. Let $\mathcal{A}$ be a weakly commutative algebra in $\mathcal{C}$. Consider the following property of $\mathcal{A}$

- ($\star$) The ring $\Gamma(\mathcal{A})$ is a finite type R-algebra and for every finite type module $\mathcal{F}$ over $\mathcal{A}$ the $\Gamma(\mathcal{A})$-module $\Gamma(\mathcal{F})$ is finite.

Lemma 9.2. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. Let $\mathcal{A} \to \mathcal{B}$ be an epimorphism of algebras in $\mathcal{C}$. Assume that*

1. *R is Noetherian and axiom (A) holds,*
2. *$\mathcal{A}$ is locally finite and F-weakly commutative, and*
3. *$\Gamma(\mathcal{B})$ is a finitely generated R-algebra.*

Then $\Gamma(\mathcal{B})$ is a finite $\Gamma(\mathcal{A})$-module and there exists a finitely generated R-subalgebra $B \subset \Gamma(\mathcal{A})$ such that

$$Im(\Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{B})) = Im(B \longrightarrow \Gamma(\mathcal{B})).$$

Proof. Since $\mathcal{A}$ is F-weakly commutative, so is $\mathcal{B}$. Hence $\Gamma(\mathcal{B})$ is a commutative R-algebra. Pick $f_1, \ldots, f_n \in \Gamma(\mathcal{B})$ which generate as an R-algebra. By axiom (A) we can find $g_1, \ldots, g_n \in \Gamma(\mathcal{A})$ which map to $f_1^{n_1}, \ldots, f_n^{n_n}$ in $\Gamma(\mathcal{B})$ for some $n_i > 0$. Then we see that $\Gamma(\mathcal{B})$ is generated by the elements

$$f_1^{e_1} \ldots f_n^{e_n}, \quad 0 \leq e_i \leq n_i - 1$$

and so $\Gamma(\mathcal{B})$ is finite over $\Gamma(\mathcal{A})$. As a first approximation, let $B = R[g_1, \ldots, g_n] \subset \Gamma(\mathcal{A})$. Then the equality of the lemma may not hold, but in any case $\Gamma(\mathcal{A})$ is finite over B. Since B is a Noetherian ring, $\mathrm{Im}(\Gamma(\mathcal{A}) \to \Gamma(\mathcal{B}))$ is a finite B-module so be choose finitely many generators $g_{n+1}, \ldots, g_{n+m} \in \Gamma(\mathcal{A})$. Hence by setting $B = R[g_1, \ldots, g_{n+m}]$, the lemma is proved. □

Lemma 9.3. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. Let $\mathcal{A}$ be an algebra in $\mathcal{C}$ and $\mathcal{I} \subset \mathcal{A}$ be a left ideal. Assume*

(1) *R is Noetherian and axiom (A) holds,*
(2) *$\mathcal{A}$ is locally finite and F-weakly commutative,*
(3) *($\star$) holds for $\mathcal{A}/\mathcal{I}$, and*
(4) *there is a quotient $\mathcal{A} \to \mathcal{A}'$ such that ($\star$) holds for $\mathcal{A}'$ and such that $\mathcal{I}$ is a finite $\mathcal{A}'$-module.*

Then ($\star$) holds for $\mathcal{A}$.

Proof. Since $\mathcal{A}$ is F-weakly commutative and locally finite so are $\mathcal{A}/\mathcal{I}$ and $\mathcal{A}'$. By Lemma 9.2 the rings $\Gamma(\mathcal{A}')$ and $\Gamma(\mathcal{A}/\mathcal{I})$ are finite $\Gamma(\mathcal{A})$-algebras. Consider the exact sequence

$$0 \to \Gamma(\mathcal{I}) \to \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A}/\mathcal{I}).$$

By ($\star$) for $\mathcal{A}'$ we see that $\Gamma(\mathcal{I})$ is a finite $\Gamma(\mathcal{A}')$-module, hence a finite $\Gamma(\mathcal{A})$-module. Choose generators $x_1, \ldots, x_s \in \Gamma(\mathcal{I})$ as a $\Gamma(\mathcal{A})$-module. By Lemma 9.2 we can find a finite type R-subalgebra $B \subset \Gamma(\mathcal{A})$ such that the image of B in $\Gamma(\mathcal{A}')$ and the image of B in $\Gamma(\mathcal{A}/\mathcal{I})$ is the same as the image of $\Gamma(\mathcal{A})$ in those rings. We claim that

$$\Gamma(\mathcal{A}) = B[x_1, \ldots, x_s]$$

as subrings of $\Gamma(\mathcal{A})$. Namely, if $h \in \Gamma(\mathcal{A})$ then we can find an element $b \in B$ which has the same image as h in $\Gamma(\mathcal{A}/\mathcal{I})$. Hence replacing h by $h - b$ we may assume $h \in \Gamma(\mathcal{I})$. By our choice of $x_1, \ldots, x_s$ we may write $h = \sum a_i x_i$ for some $a_i \in \Gamma(\mathcal{A})$. But since $\mathcal{I}$ is a $\mathcal{A}'$-module, we can write this as $h = \sum a_i' x_i$ with $a_i' \in \Gamma(\mathcal{A}')$ the image of a_i. By choice of B we can find $b_i \in B$ mapping to a_i'. Hence we see that $h \in B[x_1, \ldots, x_s]$ as desired. This proves that $\Gamma(\mathcal{A})$ is a finitely generated R-algebra.

Let $\mathcal{F}$ be a finite type $\mathcal{A}$-module. Set $\mathcal{I}\mathcal{F}$ equal to the image of the map $\mathcal{I} \otimes \mathcal{F} \to \mathcal{F}$ which is the restriction of the multiplication map of $\mathcal{F}$. Consider the exact sequence

$$0 \to \mathcal{I}\mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{I}\mathcal{F} \to 0$$

This gives rise to a similar short exact sequence on applying F, and a surjective map $F(\mathcal{I}) \otimes_A F(\mathcal{F}) \to F(\mathcal{I}\mathcal{F})$ which factors through $F(\mathcal{I}) \otimes_{F(\mathcal{A})} F(\mathcal{F})$ as $\mathcal{A}$ is F-weakly commutative. Since $F(\mathcal{F})$ is finite as a $F(\mathcal{A})$-module, and $F(\mathcal{I})$ is finite as a $F(\mathcal{A}')$-module, we conclude that $F(\mathcal{I}\mathcal{F})$ is a finite $F(\mathcal{A}')$-module, i.e., that $\mathcal{I}\mathcal{F}$ is a finite $\mathcal{A}'$-module. In the same way we see that $\mathcal{F}/\mathcal{I}\mathcal{F}$ is a finite $\mathcal{A}/\mathcal{I}$-module. Hence in the exact sequence

$$0 \to \Gamma(\mathcal{I}\mathcal{F}) \to \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}/\mathcal{I}\mathcal{F})$$

we see that the modules on the left and the right are finite $\Gamma(\mathcal{A})$-modules. Since $\Gamma(\mathcal{A})$ is Noetherian by the result of the preceding paragraph we see that $\Gamma(\mathcal{F})$ is a finite $\Gamma(\mathcal{A})$-module. This conclude the proof that property ($\star$) holds for $\mathcal{A}$. □

Lemma 9.4. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. Let $\mathcal{A}$ be an algebra in $\mathcal{C}$ and $\mathcal{I} \subset \mathcal{A}$ be a left ideal. Assume that*

1. *axioms (N) and (A) hold and R is Noetherian,*
2. *$\mathcal{A}$ is locally finite, F-weakly commutative and of finite type,*
3. *$\mathcal{I}^n = 0$ for some $n \geq 0$, and*
4. *$\mathcal{A}/\mathcal{I}$ has property $(\star)$.*

Then $\mathcal{A}$ has property $(\star)$.

Proof. We argue by induction on n and hence we may assume that $\mathcal{I}^2 = 0$. Then we get an exact sequence

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{A}/\mathcal{I} \to 0.$$

Because (N) holds and $\mathcal{A}$ is of finite type we see that $F(\mathcal{A})$ is a finitely generated A-algebra hence Noetherian. Thus $\mathcal{I}$ is a finite type $\mathcal{A}$-module, and hence also a finite type $\mathcal{A}/\mathcal{I}$-module. This means that Lemma 9.3 applies, and we win. $\square$

Theorem 9.5. *Let $(R \to A, \mathcal{C}, F)$ be as in Situation 2.1. Assume*

1. *R is Noetherian,*
2. *$R \to A$ is of finite type, and*
3. *the axioms (A) and (D) hold.*

Then for every finite type, locally finite, F-weakly commutative algebra $\mathcal{A}$ in $\mathcal{C}$ property $(\star)$ holds.

Proof. Let $\mathcal{A}$ be a finite type, locally finite, F-weakly commutative algebra $\mathcal{A}$ in $\mathcal{C}$. For every left ideal $\mathcal{I} \subset \mathcal{A}$ the quotient $\mathcal{A}/\mathcal{I}$ is also a finite type, locally finite, F-weakly commutative algebra in $\mathcal{C}$. Consider the set

$$\{\mathcal{I} \subset \mathcal{A} \mid (\star) \text{ fails for } \mathcal{A}/\mathcal{I}\}.$$

To get a contradiction assume that this set is nonempty. By Noetherian induction on the ideal $F(\mathcal{I}) \subset F(\mathcal{A})$ we see there exists a maximal left ideal $\mathcal{I}_{max} \subset \mathcal{A}$ such that $(\star)$ holds for any ideal strictly containing $\mathcal{I}_{max}$ but $(\star)$ does not hold for $\mathcal{I}_{max}$. Replacing $\mathcal{A}$ by $\mathcal{A}/\mathcal{I}_{max}$ we may assume (in order to get a contradiction) that $(\star)$ does not hold for $\mathcal{A}$ but does hold for every proper quotient of $\mathcal{A}$.

Let $f \in \Gamma(\mathcal{A})$ be nonzero. If $\mathrm{Ker}(f : \mathcal{A} \to \mathcal{A})$ is nonzero, then we see that we get an exact sequence

$$0 \to (f) \to \mathcal{A} \to \mathcal{A}/(f) \to 0$$

Since we are assuming $(\star)$ holds for both $\mathcal{A}/\mathrm{Ker}(f : \mathcal{A} \to \mathcal{A})$ and $\mathcal{A}/(f)$ and since $\mathrm{Ker}(f)$ is a finite $\mathcal{A}/(f)$-module, we can apply Lemma 9.3. Hence we see that we may assume that any nonzero element $f \in \Gamma(\mathcal{A})$ is a nonzero divisor on $\mathcal{A}$. In particular, $\Gamma(\mathcal{A})$ is a domain.

Again, assume that $f \in \Gamma(\mathcal{A})$ is nonzero. Consider the sequence

$$0 \to \mathcal{A} \xrightarrow{f} \mathcal{A} \to \mathcal{A}/f\mathcal{A} \to 0$$

which gives rise to the sequence

$$0 \to \Gamma(\mathcal{A}) \xrightarrow{f} \Gamma(\mathcal{A}) \to \mathrm{Im}(\Gamma(\mathcal{A}) \to \Gamma(\mathcal{A}/f\mathcal{A})) \to 0$$

We know that the ring on the right is a finite type R-algebra which is finite over $\Gamma(\mathcal{A})$, see Lemma 9.2. Hence any ideal $I \subset \Gamma(\mathcal{A})$ containing f maps to a finitely generated ideal in it. This implies that $\Gamma(\mathcal{A})$ is Noetherian.

Next, we claim that for any finite type $\mathcal{A}$-module $\mathcal{F}$ the module $\Gamma(\mathcal{F})$ is a finite $\Gamma(\mathcal{A})$-module. Again we can do this by Noetherian induction applied to the set

$$\{\mathcal{G} \subset \mathcal{F} \text{ is an } \mathcal{A}\text{-submodule such that finite generation fails for } \Gamma(\mathcal{F}/\mathcal{G})\}.$$

In other words, we may assume that $\mathcal{F}$ is a minimal counter example in the sense that any proper quotient of $\mathcal{F}$ gives a finite $\Gamma(\mathcal{A})$-module. Pick $s \in \Gamma(\mathcal{F})$ nonzero (if $\Gamma(\mathcal{F})$ is zero, we're done). Let $\mathcal{A} \cdot s \subset \mathcal{F}$ denote the image of $\mathcal{A} \to \mathcal{F}$ which is multiplying against s. Now we have

$$0 \to \mathcal{A} \cdot s \to \mathcal{F} \to \mathcal{F}/\mathcal{A} \cdot s \to 0$$

which gives the exact sequence

$$0 \to \Gamma(\mathcal{A} \cdot s) \to \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}/\mathcal{A} \cdot s)$$

By minimality we see that the module on the right is finite over the Noetherian ring $\Gamma(\mathcal{A})$. On the other hand, the module on the left is $\Gamma(\mathcal{A}/\mathcal{I})$ for the ideal $\mathcal{I} = \mathrm{Ker}(s : \mathcal{A} \to \mathcal{F})$. If $\mathcal{I} = 0$ then this is $\Gamma(\mathcal{A})$ and therefore finite, and if $\mathcal{I} \neq 0$ then this is a finite $\Gamma(\mathcal{A})$-module by Lemma 9.2 and minimality of $\mathcal{A}$. Hence we conclude that the middle module is finite over the Noetherian ring $\Gamma(\mathcal{A})$ which is the desired contradiction.

Finally, we show that $\Gamma(\mathcal{A})$ is of finite type over R which will finish the proof. Namely, by Lemma 8.4 the morphism of schemes

$$\mathrm{Spec}(F(\mathcal{A})) \longrightarrow \mathrm{Spec}(\Gamma(\mathcal{A}))$$

is universally submersive. We have already seen that $\Gamma(\mathcal{A})$ is a Noetherian ring. Thus Theorem 8.5 kicks in and we are done. $\square$

10. Quasi-coherent sheaves on algebraic stacks

Let $S = \mathrm{Spec}(R)$ be an affine scheme. Let $\mathcal{X}$ be a quasi-compact algebraic stack over S. Let $p : T \to \mathcal{X}$ be a smooth surjective morphism from an affine scheme $T = \mathrm{Spec}(A)$.

Lemma 10.1. *In the situation above, the category $QCoh(\mathcal{O}_\mathcal{X})$ endowed with its natural tensor product, pullback functor $F : QCoh(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_T) = Mod_A$ and structure sheaf $\mathcal{O} = \mathcal{O}_\mathcal{X}$ is an example of Situation 2.1. The functor $\Gamma : QCoh(\mathcal{O}_\mathcal{X}) \to Mod_R$ is identified with the functor of global sections*

$$\mathcal{F} \longmapsto \Gamma(\mathcal{X}, \mathcal{F}).$$

Axioms (D), (C), and (S) hold. If $\mathcal{X}$ is noetherian (e.g., $\mathcal{X}$ is quasi-separated and A is Noetherian), then axiom (L) holds.

Proof. The final statement is [LMB00, Prop 15.4]. The rest is clear. $\square$

The following definition reinterprets the adequacy axiom (A).

Definition 10.2. Let $\mathcal{X}$ be an quasi-compact algebraic stack over $S = \mathrm{Spec}(R)$. We say that $\mathcal{X}$ is *adequate* if for every surjection $\mathcal{A} \to \mathcal{B}$ of quasi-coherent $\mathcal{O}_\mathcal{X}$-algebras with $\mathcal{A}$ locally finite and $f \in \Gamma(\mathcal{X}, \mathcal{B})$, there exists an $n > 0$ and a $g \in \Gamma(\mathcal{X}, \mathcal{A})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{X}, \mathcal{B})$.

Lemma 10.3. *Let $\mathcal{X}$ be an quasi-compact algebraic stack over $S = Spec(R)$. The following are equivalent:*

(1) *$\mathcal{X}$ is adequate.*

(2) *For every surjection of finite type $\mathcal{O}_\mathcal{X}$-modules $\mathcal{G} \to \mathcal{F}$ and $f \in \Gamma(\mathcal{X}, \mathcal{F})$, there exists an $n > 0$ and a $g \in \Gamma(\mathcal{X}, Sym^n \mathcal{G})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{X}, Sym^n \mathcal{F})$.*

If $\mathcal{X}$ is noetherian, then the above are also equivalent to:

(3) *For every surjection $\mathcal{G} \to \mathcal{O}$ with $\mathcal{G}$ of finite type and $f \in \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$, there exists an $n > 0$ and a $g \in \Gamma(\mathcal{X}, Sym^n \mathcal{G})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$.*

(1') *For every surjection $\mathcal{A} \to \mathcal{B}$ of quasi-coherent $\mathcal{O}_\mathcal{X}$-algebras and $f \in \Gamma(\mathcal{X}, \mathcal{B})$, there exists an $n > 0$ and a $g \in \Gamma(\mathcal{X}, \mathcal{A})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{X}, \mathcal{B})$.*

(2') *For every surjection of $\mathcal{O}_\mathcal{X}$-modules $\mathcal{G} \to \mathcal{F}$ and $f \in \Gamma(\mathcal{X}, \mathcal{F})$, there exists an $n > 0$ and a $g \in \Gamma(\mathcal{X}, Sym^n \mathcal{G})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{X}, Sym^n \mathcal{F})$.*

(3') *For every surjection $\mathcal{G} \to \mathcal{O}$ and $f \in \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$, there exists an $n > 0$ and a $g \in \Gamma(\mathcal{X}, Sym^n \mathcal{G})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$.*

Proof. This follows from Lemma 7.7. $\square$

Corollary 10.4. *Let $\mathcal{X}$ be an algebraic stack finite type over an affine noetherian scheme Spec(R). Suppose $\mathcal{X}$ is adequate. Let $\mathcal{A}$ be a finite type $\mathcal{O}_\mathcal{X}$-algebra. Then $\Gamma(\mathcal{X}, \mathcal{A})$ is finitely generated over R and for every finite type $\mathcal{A}$-module $\mathcal{F}$, the $\Gamma(\mathcal{X}, \mathcal{A})$-module $\Gamma(\mathcal{X}, \mathcal{F})$ is finite.*

Proof. This follows from Theorem 9.5. $\square$

11. Bialgebras, modules and comodules

In this section we discuss how modules and comodules over a bialgebra form an example of our abstract setup. If R is a commutative ring, recall that a *bialgebra H over R* is an R-module H endowed with maps $(R \to H, H \otimes_R H \to R, \epsilon : H \to R, \delta : H \to H \otimes_R H)$. Here $H \otimes_R H \to H$ and $R \to H$ define an unital R-algebra structure on H, the maps δ and ϵ are unital R-algebra maps. Moreover, the comultiplication μ is associative and ϵ is a counit.

Let H be a bialgebra over R. A *left H-module* is a left module over the R-algebra structure on H; that is, there is a R-module homomorphism $H \otimes_R M \to M$ satisfying the two commutative diagrams for an action. A *left H-comodule* M is an R-module homomorphism $\sigma : M \to H \otimes_R M$ satisfying the two commutative diagram for a coaction. See [Kas95, Chapter 3] and [Mon93, Chapter 1] for the basic properties of H-modules and H-comodules.

Definition 11.1. Let R be a commutative ring. Let H be a bialgebra over R.

(1) Let Mod_H be the category of left H-modules. It is endowed with the forgetful functor to R-modules, the tensor product

$$(M, N) \longmapsto M \otimes_R N$$

where H acts on $M \otimes_R N$ via the comultiplication, and the object $\mathcal{O}$ given by the module R where H acts via the counit.

(2) Let Comod_H be the category of left H-comodules. It is endowed with the forgetful functor to R-modules, the tensor product

$$(M, N) \longmapsto M \otimes_R N$$

where the comodule structure on $M \otimes_R N$ comes from the multiplication in H, and the object $\mathcal{O}$ given by the module R where H acts via the R-algebra structure H.

Lemma 11.2. *Let R be a commutative ring and H be a bialgebra over R.*

(1) *The category Mod_H with its additional structure introduced in Definition 11.1 is an example of Situation 2.1. The functor $\Gamma : Mod_H \to Mod_R$ is identified with the functor of invariants*
$$M \longmapsto M^H = \{m \in M \mid h \cdot m = \epsilon(h)m\}.$$
Axioms (D), (I) and (S) hold. Axiom (C) holds if H is cocommutative.

(2) *The category $Comod_H$ with its additional structure introduced in Definition 11.1 is an example of Situation 2.1. The functor $\Gamma : Comod_H \to Mod_R$ is identified with the functor of coinvariants*
$$M \longmapsto M_H = \{m \in M \mid \sigma(m) = 1 \otimes m\}$$
where $\sigma : M \to H \otimes_R M$ indicates the coaction of M. Axiom (D) holds. Axiom (C) holds if H is commutative. If R is a field, then axiom (L) holds.

Proof. The first two statements in both part (1) and (2) are clear. It also clear that axiom (D) holds in both cases. Arbitrary direct products exist in the category Mod_H, which is axiom (I), and so by Lemma 4.8 axiom (S) holds. The statement concerning axiom (C) is straightforward; see [Mon93, Section 1.8]. It is well-known that axiom (L) holds for Comod_H. $\square$

12. Adequacy for a bialgebra

Let R be map of commutative rings. Let H be a bialgebra over R. Let M be an H-module. We can identify $\mathrm{Sym}^n_H M := \mathrm{Sym}^n_{\mathrm{Mod}_H} M$ of axiom (S) with the H-module
$$\underbrace{M \otimes_R \cdots \otimes_R M}_{n} / M'$$
where M' is the submodule generated by elements $h\cdot((\cdots\otimes m_i \otimes\cdots\otimes m_j\otimes\cdots) - (\cdots\otimes m_j\otimes\cdots\otimes m_i\otimes\cdots))$ for $h \in H$ and $m_1, \ldots, m_n \in M$. And $\mathrm{Sym}_H M := \bigoplus_n \mathrm{Sym}^n_H M$ is the largest H-module quotient of the tensor algebra on M which is commutative.

Example 12.1. We provide an example of a non-cocommutative bialgebra together with a description of certain symmetric products. Let G be a non-abelian finite group and k be a field. The dual $H = k[G]^*$ inherits a natural structure of a bialgebra from the bialgebra structure on $k[G]$. Explicitly, if $\{p_g \mid g \in G\}$ denotes a dual basis for $k[G]^*$, then multiplication is given by $m(p_g \otimes p_h) = p_g$ if $g = h$ and 0 otherwise, and comultiplication is given by $\delta(p_g) = \sum_{uv=g} p_u \otimes p_v$. Axiom (C) does not hold; for instance, $k\langle p_g\rangle \otimes k\langle p_h\rangle \cong k\langle p_{gh}\rangle$ which is not isomorphic to $k\langle p_h\rangle \otimes k\langle p_g\rangle$ if $gh \neq hg$. Let $M = k[G]^*$ with the H-module structure given by multiplication. Then one checks that
$$\begin{aligned}\mathrm{Sym}^n_H M =& \Big\langle p_{g_1} \otimes \cdots \otimes p_{g_n} \mid g_i \in G \text{ such that } \forall \sigma \in S_n, \prod_i g_i = \prod_i g_{\sigma(i)} \Big\rangle \Big/ \\ & \Big\langle (\cdots \otimes p_{g_i} \otimes \cdots \otimes p_{g_j} \otimes \cdots) - (\cdots \otimes p_{g_j} \otimes \cdots \otimes p_{g_i} \otimes \cdots) \Big\rangle.\end{aligned}$$

An *H-module algebra* is an H-module C which is an algebra over the algebra structure on H such that $R \to C$ and $C \otimes_R C \to C$ are H-module homomorphisms. We say that C is *commutative* if C is commutative as an algebra. An H-module M is *locally finite* if it is the filtered colimit of finite type H-modules.

The following definition reinterprets adequacy axiom (A) for the category Mod_H.

Definition 12.2. Let R be a commutative ring and H be a bialgebra over R. We say that H is *adequate* if for every surjection of commutative H-module algebras $C \to D$ in Mod_H with C locally finite, and any $f \in D^H$, there exists an $n > 0$ and an element $g \in C^H$ such that $g \mapsto f^n$ in D^H.

Lemma 12.3. *Let R be a commutative ring and H be a bialgebra over R. The following are equivalent:*

(1) *H is adequate.*
(2) *For every surjection of finite type H-modules $N \to M$ and $f \in M^H$, there exists an $n > 0$ and a $g \in (Sym_H^n N)^H$ such that $g \mapsto f^n$ in $(Sym_H^n M)^H$.*

If A is Noetherian, then the above are also equivalent to:

(3) *For every surjection of finite type H-modules $N \to R$ and $f \in R$, there exists an $n > 0$ and a $g \in (Sym_H^n N)^H$ such that $g \mapsto f^n$ in R.*

Proof. This follows from Lemma 7.7. □

Corollary 12.4. *Let R be a commutative Noetherian ring and H be an adequate bialgebra over R. Let C be a finitely generated, locally finite, commutative H-module algebra. Then C^H is a finitely generated R-algebra and for every finite type C-module M, the C^H-module M^H is finite.*

Proof. This follows from Theorem 9.5. □

Remark 12.5. If $R = k$ is a field, then [KT08] define a Hopf algebra H over k to be *geometrically reductive* if any finite dimensional H-module M and any non-zero homomorphism of H-modules $N \to k$ there exist $n > 0$ such that $\mathrm{Sym}_H^n(N)^H \to k$ is non-zero. By Lemma 12.3, H is geometrically reductive if and only if H is adequate.

In [KT08, Theorem 3.1], Kalniuk and Tyc prove that with the hypotheses of the above corollary and with the additional assumption that R is a field, then C^H is finitely generated over R.

13. Coadequacy for a bialgebra

Let R be a commutative ring. Let H be a bialgebra over R. An *H-comodule algebra* is an H-comodule C which is an algebra over the algebra structure on H such that $R \to C$ and $C \otimes_R C \to C$ are H-comodule homomorphisms; C is *commutative* if C is commutative as an algebra. An H-comodule M is *locally finite* if it is the filtered colimit of finite type H-comodules.

Here we reinterpret the adequacy axiom (A) for the category Comod_H.

Definition 13.1. Let R be a commutative ring and H be a bialgebra over R. We say that H is *coadequate* if for every surjection of commutative H-comodule algebras $C \to D$ with C locally finite, and any $f \in D_H$, there exists an $n > 0$ and an element $g \in C_H$ such that $g \mapsto f^n$ in D_H.

Recall that we only know that axiom (S) holds for Comod_H when H is commutative.

Lemma 13.2. *Let R be a commutative ring and H be a commutative bialgebra over R. The following are equivalent:*

(1) *H is coadequate.*

(2) *For every surjection of finite type H-modules $N \to M$ and $f \in M^H$, there exists an $n > 0$ and a $g \in (Sym_H^n N)^H$ such that $g \mapsto f^n$ in $(Sym_H^n M)^H$.*

If A is Noetherian, then the above are also equivalent to:

(3) *For every surjection of finite type H-modules $N \to R$ and $f \in R$, there exists an $n > 0$ and a $g \in (Sym_H^n N)^H$ such that $g \mapsto f^n$ in R.*

Proof. This follows from Lemma 7.7. □

Corollary 13.3. *Let R be a commutative Noetherian ring and H be a coadequate bialgebra over R. Let C be a finitely generated, locally finite, commutative H-comodule algebra. Then C_H is a finitely generated R-algebra and for every finite type C-module M, the C_H-module M_H is finite.*

Proof. This follows from Theorem 9.5. □

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