Theorem (Theorem 9.19, Exercise 9D). Suppose $ABCD$ is a quadrilateral.

(a) If its diagonals bisect each other, then $ABCD$ is a parallelogram.

(b) If its diagonals are congruent and bisect each other, then $ABCD$ is equiangular.

(c) If its diagonals are perpendicular bisectors of each other, then $ABCD$ is a rhombus.

(d) If its diagonals are congruent and are perpendicular bisectors of each other, then $ABCD$ is a regular quadrilateral.

Proof. (a) Suppose that $ABCD$ is a quadrilateral in which the diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other. Let $O$ be the point of intersection.

Then the triangles $\triangle COB$ and $\triangle AOD$ are congruent: we have $\overline{AO} \cong \overline{CO}$ and $\overline{BO} \cong \overline{DO}$ because $O$ bisects each of the diagonals. The angles $\angle AOD$ and $\angle COB$ are congruent because they are vertical angles. So by SAS, we have $\triangle COB \cong \triangle AOD$. Therefore $\overline{BC} \cong \overline{AD}$. A similar argument shows that $\overline{AB} \cong \overline{CD}$.

This means that the pairs of opposite sides are congruent, so by Theorem 9.17, $ABCD$ is a parallelogram.

(b) Suppose that $ABCD$ is a quadrilateral in which the diagonals $\overline{AC}$ and $\overline{BD}$ are congruent and bisect each other. Let $O$ be the point of intersection. Note that the diagonal criterion for convex quadrilaterals tells us that $ABCD$ is convex.

Then the triangles $\triangle COB$ and $\triangle AOD$ are congruent, by the same proof as in part (a); furthermore, they are both isosceles. Therefore $\angle OBC \cong \angle OCB$ and $\angle OAD \cong \angle ODA$. Similarly, $\angle COD \cong \angle AOB$ and both are isosceles, so $\angle OCD \cong \angle ODC$ and $\angle OBA \cong \angle OAB$. Since the point $O$ is in the interior of the quadrilateral (by Theorem 9.4(b), the diagonal criterion for convex quadrilaterals), the ray $\overrightarrow{BO}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, so $m\angle ABC = m\angle ABO + m\angle OBC$. (Alternatively, appeal to Lemma 9.3.) Since $\angle ABO \cong \angle BAO$ and $\angle OBC \cong \angle ODA \cong \angle OAD$, we see that

\[
m\angle ABC = m\angle ABO + m\angle OBC = m\angle BAO + m\angle OAD = m\angle BAD.
\]
A similar computation shows that all four angles of the quadrilateral have equal measure. Since it is convex, it is equiangular.

(c) Suppose that $ABCD$ is a quadrilateral in which the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular bisectors of each other. Let $O$ be the point of intersection.

Now all four proper angles at the point $O$ are right angles (by the four right angles theorem). I claim that the four triangles $\triangle AOB$, $\triangle COB$, $\triangle COD$, and $\triangle AOD$ are congruent to each other by SAS. For example, $\angle AOB \cong \angle COB$, $\overline{AO} \cong \overline{CO}$, and $\overline{BO} \cong \overline{BO}$, so $\triangle AOB \cong \triangle COB$. Similar arguments show that $\triangle AOB$ is congruent to the other two triangles. Therefore we have congruences of sides $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{AD}$.

So the quadrilateral is a rhombus.

(d) Suppose that $ABCD$ is a quadrilateral in which the diagonals are congruent and are perpendicular bisectors of each other. Then by part (b), the quadrilateral is equiangular, and by part (c), it is a rhombus: it is equilateral. By definition, a polygon is regular if it is equilateral and equiangular; therefore $ABCD$ is regular.

**Corollary** (Corollary 10.2, Converse to the Corresponding Angles Theorem, Exercise 10A). *If two parallel lines are cut by a transversal, then all four pairs of corresponding angles are congruent.*

**Proof.** Suppose that $\ell$ and $\ell'$ are parallel, cut by the transversal $t$. Suppose that $t$ intersects $\ell$ at $A$ and $\ell'$ at $A'$. Choose one pair of corresponding angles and choose points $B \in t$, $P \in \ell$, and $P' \in \ell'$ such that the angles are $\angle BAP$ and $\angle BA'P'$. We need to show that $\angle BAP \cong \angle BA'P'$.

Choose a point $Q$ on $\ell$ so that $Q \ast A \ast P$. Then $\angle QAA'$ and $\angle AA'P'$ are a pair of alternating interior angles, so they are congruent by the converse to the alternate interior angles theorem. Since $\angle QAA'$ and $\angle BAP$ are vertical angles, they are congruent, and therefore $\angle BAP \cong \angle AA'P'$, as desired.

**Theorem** (Theorem 10.10, Transitivity of Parallelism, Exercise 10D). *If $\ell$, $m$, and $n$ are distinct lines such that $\ell \parallel m$ and $m \parallel n$, then $\ell \parallel n$.*

**Proof.** If $\ell$ and $m$ are parallel, then they are equidistant (by Theorem 10.8, the converse to the equidistance theorem); similarly, if $m$ and $n$ are parallel, then they are equidistant. Therefore $\ell$ and $n$ are equidistant, so by the equidistance theorem (7.24), $\ell$ and $n$ are parallel.
Theorem (Theorem 10.14, 30-60-90 Theorem, Exercise 10F). A triangle has interior angle measures 30°, 60°, and 90° if and only if it is a right triangle in which the hypotenuse is twice as long as the shorter leg.

Proof. Let \( \triangle ABC \) be a triangle with \( m\angle A = 30^\circ \), \( m\angle B = 60^\circ \), and \( m\angle C = 90^\circ \). Then certainly \( \triangle ABC \) is a right triangle. Also, the hypotenuse is \( \overline{AB} \) and the shorter leg is \( \overline{BC} \), by the scalene inequality. So we need to show that \( AB = 2BC \). By the triangle copying theorem, there is a point \( D \) on the opposite side of \( \overline{AC} \) from \( B \) so that \( \triangle ACB \cong \triangle ACD \). This means that \( \angle ACB \cong \angle ACD \), so these are both right angles, and therefore are adjacent supplementary angles. This means that they form a linear pair, by Theorem 4.17 (Partial Converse to the Linear Pair Theorem), so \( B \), \( C \), and \( D \) are collinear with \( B \neq C \neq D \). The triangle \( \triangle ABD \) is therefore a 60-60-60 triangle, and \( C \) is the midpoint of \( BD \). Therefore \( 2BC = BD = AB \), which says that the hypotenuse is twice as long as the shorter leg.

Conversely, suppose that \( \triangle ABC \) is a right triangle in which the hypotenuse is twice as long as the shorter leg. Without loss of generality, suppose that \( \angle C \) is a right angle and \( \overline{BC} \) is the shorter leg; thus we have \( 2BC = AB \). Use the triangle copying theorem again, to get a point \( D \) on the other side of \( \overline{AC} \) from \( B \), so that \( \triangle ABC \cong \triangle ACD \). Then, as in the first part of the proof, \( \angle ACB \) and \( \angle ACD \) are supplementary and adjacent, so they form a linear pair, so \( B \neq C \neq D \). Thus \( \triangle ABD \) is a triangle in which \( AB = AD = 2BC = BC + CD = BD \).

That is \( \triangle ACD \) is an equilateral triangle. By the 60-60-60 Theorem, every angle must be 60°, and in particular, \( m\angle B = 60^\circ \). By the Angle-Sum Theorem for Triangles,

\[
m\angle A = 180^\circ - m\angle C - m\angle B = 180^\circ - 90^\circ - 60^\circ = 30^\circ.
\]

So \( \triangle ABC \) has angle measures 30°, 60°, and 90°. \( \square \)