Garfield’s proof of the Pythagorean theorem, Exercise 13A. Here is the picture:

We start with a right triangle \( \triangle ABC \) with right angle \( \angle C \), leg lengths \( a \) and \( b \) (for the legs opposite vertices \( A \) and \( B \), respectively), hypotenuse \( c \). We want to prove that \( c^2 = a^2 + b^2 \).

First extend the segment \( AC \) to have length \( a + b \); in the picture, this means that \( C, A, \) and \( D \) are collinear, and \( AD \) has length \( a \). Now construct the perpendicular to \( AD \) at the point \( D \) (by constructing a perpendicular, 4.32), and then on that perpendicular, find the point \( E \) on the same side of \( AD \) as \( B \), so that \( DE \) has length \( b \). Then \( \triangle ADE \) is a right triangle which is congruent to \( \triangle BCA \), so its hypotenuse has length \( c \).

Using the linear triple theorem and algebra, we can see that \( m\angle BAE = 90^\circ \), so \( \triangle ABE \) is a right triangle with both legs of length \( c \).

Now we compute the area of the trapezoid \( BCDE \) in two different ways. On one hand, by Theorem 11.13, the area is

\[
\frac{1}{2}(a + b)(a + b),
\]

since the “height” (measured from left to right in this picture) is \( a + b \), and the two bases have lengths \( a \) and \( b \). On the other hand, we can decompose the trapezoid into non-overlapping triangles \( \triangle ABC, \triangle ABE, \) and \( \triangle ADE \). So the area of the trapezoid is the sum of the areas of these triangles: the area is

\[
\frac{1}{2}ab + \frac{1}{2}c^2 + \frac{1}{2}ab.
\]

Equating these and doing some algebra gives the desired result.

\[\square\]

Proof of the Pythagorean theorem, Exercise 13B. Suppose given a right triangle \( \triangle ABC \) with hypotenuse of length \( c \), legs of lengths \( a \) and \( b \) with \( a \leq b \).

If \( a = b \), then we find a point \( Y \) on \( AC \) so that \( AY = 2AC \). Then \( CY = CB = AC \); also, since \( \angle C = \angle BCA \) is right, so is \( \angle BCY \) (by the linear pair theorem), so \( \triangle BCY \) is a 45-45-90 isosceles right triangle. Repeat the construction to get a point \( Z \) so that \( \triangle CYZ \) is a 45-45-90 triangle; then an easy computation shows that \( \triangle CAZ \) is also 45-45-90. We end up with this situation:
The points $A$, $C$, and $Y$ are collinear, by construction, and so are the points $B$, $C$, and $Z$. So by the convex decomposition lemma, the area of the square is the sum of the areas of $\triangle ABY$ and $\triangle AYZ$; these in turn are given by

\[
\alpha(\triangle ABY) = \alpha(\triangle ACB) + \alpha(\triangle BCY), \\
\alpha(\triangle AYZ) = \alpha(\triangle ACY) + \alpha(\triangle CYZ).
\]

The large square has area $c^2$, and each triangle has area $\frac{1}{2}ab = \frac{1}{2}a^2$, so we get

\[
c^2 = 2a^2 = a^2 + b^2.
\]

Now suppose instead that $a < b$. Construct a perpendicular to $\overrightarrow{AB}$ at $B$, and construct a segment $\overrightarrow{BY}$ of that perpendicular of length $c$ on the same side $\overrightarrow{AB}$ as $C$ is. Then a simple computation shows that $\angle YBX$ and $\angle ABC$ are complementary, so $B$, $C$ and $X$ are collinear. Repeat this, for example, at the point $Y$, to get a point $Z$ so that $ABYZ$ is a square with side length $c$, and a point $W$ as in the diagram, so that $\triangle ABC$, $\triangle BYX$, and $\triangle YZW$ are all congruent. The ray $\overrightarrow{ZW}$ intersects the line $\overrightarrow{AC}$ somewhere, and we let $D$ be the point of intersection. Consider $\triangle AZD$: the angles $\angle DAZ$ and $\angle DAB$ are complementary, as are $\angle AZD$ and $\angle YZW$. Therefore some algebra show that $\angle DAZ$ and $\angle AZD$ are complementary; this means that $\angle ADZ$ is right, so $\triangle ADZ$ is a right triangle.

By SAS (using the side $\overrightarrow{AZ}$), we see that it is also congruent to $\triangle ABC$. So we have the following diagram.

Then the area of the large square is $c^2$. The four congruent triangles, by construction and computation, do not overlap, and this is also true for the quadrilateral $DCXW$. So we have

\[
\alpha(ABYZ) = 4\alpha(\triangle ABC) + \alpha(DCXW).
\]

Each triangle has area $\frac{1}{2}ab$, and we claim that $DCXW$ is a square with side length $b - a$. Given this, we get

\[
c^2 = 4\left(\frac{1}{2}ab\right) + (b - a)^2 = 2ab + (b - a)^2 = a^2 + b^2,
\]

as desired.

It remains to verify that the quadrilateral is a square with side length $b - a$. We know that $B * C * X$ and that $BX = b$ while $BC = a$, so simple algebra shows that $CX = b - a$. This is true for each side of the
square, by the same argument. Since \( \angle ACB \) is right, so is \( \angle ACX \), by the linear pair theorem. Therefore each side of \( DCXW \) has the same length, namely \( b - a \), and each angle is a right angle. Thus \( DCXW \) is a square, and this finishes the proof.

**Theorem** (Theorem 13.2, Converse to the Pythagorean Theorem, Exercise 13C). \( \triangle ABC \) is a triangle with side lengths \( a \), \( b \), and \( c \). If \( a^2 + b^2 = c^2 \), then \( \triangle ABC \) is a right triangle, and its hypotenuse is the side of length \( c \).

**Proof.** Given \( \triangle ABC \) with side lengths satisfying \( a^2 + b^2 = c^2 \) as in the statement, construct a right triangle \( \triangle DEF \) with legs of length \( a \) and \( b \), and let \( x \) be the length of the hypotenuse. Then by the Pythagorean theorem, \( x^2 = a^2 + b^2 \), so \( x^2 = c^2 \). Since \( x \) and \( c \) are both positive (being the lengths of segments), we must have \( x = c \). Therefore by SSS, the triangles \( \triangle ABC \) and \( \triangle DEF \) are congruent, so \( \triangle ABC \) is a right triangle and its hypotenuse has length \( x = c \).

**Theorem** (Theorem 13.13, Law of Sines, Exercise 13H). Let \( \triangle ABC \) be any triangle, and let \( a \), \( b \), and \( c \) denote the lengths of the sides opposite \( A \), \( B \), and \( C \), respectively. Then

\[
\frac{\sin \angle A}{a} = \frac{\sin \angle B}{b} = \frac{\sin \angle C}{c}.
\]

**Proof.** Without loss of generality, it suffices to prove

\[
\frac{\sin \angle A}{a} = \frac{\sin \angle B}{b}.
\]

There are three cases: (1) \( \angle A = 90^\circ \), (2) \( \angle A \) and \( \angle B \) both acute, (3) \( \angle A \) obtuse and \( \angle B \) acute.

For case (1), we have this diagram:

\[
\begin{array}{c}
\text{C} \\
\downarrow b \\
\text{A} \quad \text{c} \\
\downarrow x \\
\text{B}
\end{array}
\]

From (13.12), \( \sin \angle A = 1 \), so

\[
\frac{\sin \angle A}{a} = \frac{1}{a}.
\]

From the definition of sine, \( \sin \angle B = b/a \), so

\[
\frac{\sin \angle B}{b} = \frac{b}{a} \cdot \frac{1}{b} = \frac{1}{a},
\]

as desired.

For case (2), we have this diagram:

\[
\begin{array}{c}
\text{C} \\
\downarrow b \\
\text{A} \quad \text{x} \\
\downarrow c \\
\text{B}
\end{array}
\]

Draw an altitude from \( C \) to \( \overrightarrow{AB} \) and let \( x \) be its length; since \( \angle A \) and \( \angle B \) are acute, this altitude intersects \( \overrightarrow{AB} \) in the interior of \( \overrightarrow{AB} \). Then we have \( \sin \angle A = x/b \) and \( \sin \angle B = x/a \), so

\[
\frac{\sin \angle A}{a} = \frac{x}{ba} \quad \text{and} \quad \frac{\sin \angle B}{b} = \frac{x}{ab}.
\]

These are equal.
For case (3), we have this picture:

\[ \begin{array}{c}
\text{C} \\
\text{a} \\
b \\
\text{b} \\
\text{x} \\
\text{A} \\
c \\
\text{B} \\
\end{array} \]

Drop a perpendicular from C to \( \overrightarrow{AB} \) and let x be its length; since \( \angle A \) is obtuse, this perpendicular intersects \( \overrightarrow{AB} \) at a point F with \( F \neq A \neq B \). Then we have \( \sin \angle A = x/b \) and \( \sin \angle B = x/a \), so

\[
\frac{\sin \angle A}{a} = \frac{x}{ba} \text{ and } \frac{\sin \angle B}{b} = \frac{x}{ab}.
\]

These are equal.

**Theorem** (Theorem 13.14, Heron’s formula, Exercise 13I). Let \( \triangle ABC \) be a triangle, and let \( a, b, \) and \( c \) denote the lengths of the sides opposite \( A, B, \) and \( C \), respectively. Then

\[
\alpha(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)},
\]

where \( s = (a+b+c)/2 \) (the semiperimeter of \( \triangle ABC \)).

**Proof.** First of all, it will be helpful to expand the right side of Heron’s formula, so that we know what we’re trying to prove. If we plug in \( s = (a+b+c)/2 \) and multiply out the right side, we get

\[
\frac{1}{4} \sqrt{-a^4 - b^4 - c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2}.
\]

We want to show that this equals the area of a triangle with side lengths \( a, b, \) and \( c \).

Here is a diagram.

\[ \begin{array}{c}
\text{C} \\
\text{b} \\
h \\
\text{a} \\
\text{y} \\
\text{A} \\
x \\
\text{B} \\
\end{array} \]

For one proof, we assume that \( \overrightarrow{AB} \) is the longest edge: assume that \( c \geq a, b \). Let \( CF \) be an altitude from \( C \) to \( \overrightarrow{AB} \); since \( AB \) is the longest edge, the point \( F \) is in the interior of \( AB \). Let \( y = AF \), let \( x = BF \), and let \( h = CF \), as in the picture. By the computations in the proof of Theorem 13.6, we have

\[
x = \frac{c^2 - b^2 + a^2}{2c}, \quad y = \frac{c^2 + b^2 - a^2}{2c}, \quad h = \sqrt{a^2 - x^2}.
\]

The area of the triangle is then

\[
\alpha(\triangle ABC) = \frac{1}{2} hc = \frac{1}{2} \sqrt{a^2 - x^2} c
\]

\[
= \frac{1}{4} (a^2 c^2 - x^2 c^2)
\]

\[
= \frac{1}{4} \left( a^2 c^2 - \frac{1}{4} (c^2 - b^2 + a^2)^2 \right)
\]

\[
= \frac{1}{4} \sqrt{4a^2c^2 - (a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 + 2a^2c^2)}
\]

\[
= \frac{1}{4} \sqrt{-a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2a^2c^2},
\]

\[ \text{(4)} \]
and this is the formula we want.

For a second proof, we use the same diagram. From the diagram, we have \( \cos \angle A = \frac{y}{b} = \sqrt{1 - \frac{h^2}{b^2}} \), so the law of cosines (applied to the angle \( \angle A \)) says that

\[
b^2 + c^2 = a^2 + 2bc \sqrt{1 - \frac{h^2}{b^2}}.
\]

Square both sides and solve for \( h \) to get

\[
h = b \sqrt{1 - \left( \frac{b^2 + c^2 - a^2}{2bc} \right)^2}.
\]

So the area of the triangle is

\[
\alpha(\triangle ABC) = \frac{1}{2} \cdot h \cdot c
= \frac{1}{2} \cdot b \cdot c \sqrt{1 - \left( \frac{b^2 + c^2 - a^2}{2bc} \right)^2}
= \frac{1}{2} \cdot b \cdot c \sqrt{\frac{4b^2c^2}{4b^2c^2} - \frac{(b^2 + c^2 - a^2)^2}{4b^2c^2}}
= \frac{1}{4} \sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2},
\]

and by algebra similar to that in the first proof, this turns into the formula we want. \( \square \)