Selected solutions for Math 445 HW7, Summer 2013

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**Theorem** (Theorem 12.8, Median Concurrence Theorem, Exercise 12I). The medians of a triangle are concurrent, and the distance from the point of intersection to each vertex is twice the distance to the midpoint of the opposite side.

**Proof.** Following the hint, let $BM$ and $CN$ be two of the medians of $\triangle ABC$. By the midsegment theorem (10.29), $\overrightarrow{BC}$ and $\overrightarrow{MN}$ are parallel, so $BCMN$ is a trapezoid, and by Corollary 9.6, $BCMN$ must be convex. Therefore its diagonals $BM$ and $CN$ intersect somewhere, by the diagonal criterion for convexity (Theorem 9.4(b)). Let $X$ be the point of intersection of $BM$ and $CN$.

Then $\angle MXN \cong \angle BXC$, since they form a pair of vertical angles. Also, since $\overrightarrow{MN}$ and $\overrightarrow{BC}$ are parallel, $\angle NMB \cong \angle CBM$ and $\angle CNM \cong \angle NCB$. Therefore $\triangle MNX \sim \triangle BCX$. By the midsegment theorem again, we know that $MN/BC = 1/2$, and therefore $MX/XB = 1/2$, which means that $2MX = XB$, or equivalently, $MX = \frac{1}{3}BM$.

We can repeat this argument with the $BM$ and the third median, say $AL$. We conclude that they intersect at a point $Y$ so that $MY = \frac{1}{3}BM$. But by the uniqueness clause in the segment construction theorem (applied to $\overrightarrow{MB}$ and the number $\frac{1}{3}BM$), this means that $X = Y$. So $BM$, $CN$, and $AL$ all intersect at the same point. The proof that $2MX = XB$ applies equally well to the other medians, thus proving the theorem.

**Theorem** (Theorem 14.30, the incircle theorem, Exercise 14O). A polygon $P$ is tangential if and only if it is convex and the bisectors of all of its angles are concurrent. If this is the case, the point $O$ where these bisectors intersect is the unique incenter for $P$, and the circle with center $O$ and radius equal to the distance from $O$ to the line containing any edge is the unique incircle.

**Proof.** First assume that $P$ is tangential. Then by Theorem 14.28, it must be convex. Let $\mathcal{C} = \mathcal{C}(O,r)$ be an circle for $P$. Then every edge of $P$ is tangent to $\mathcal{C}$. Let $A$ be a vertex of the polygon, and let $P$ and $Q$ be the points on the adjacent edges where they are tangent to the circle.

![Diagram](attachment:image.png)

Then $OQ \perp OP$, and $\angle OQA$ and $\angle OPA$ are right triangles. The triangles $\triangle AQP$ and $\triangle APO$ are right triangles which share the hypotenuse and have one pair of congruent legs, so they are congruent; therefore $\angle AQP \cong \angle PAO$ and $\overrightarrow{AO}$ is the angle bisector. Thus the center $O$ of the circle is on every angle bisector: the angle bisectors are concurrent.

Conversely, suppose that $P$ is convex and its angle bisectors are concurrent at some point $O$. By the angle bisector theorem (7.15), $O$ is equidistant from each ray forming the angle; thus if we let $P_1$, $P_2$, $\ldots$,
$P_n$ be the points on the edges of the polygon which are closest to $O$, then $OP_i$ is the same for every $i$. Let $r = OP_i$; then every point $P_i$ is on the circle $c(O, r)$. Since each segment $OP_i$ is perpendicular to the corresponding edge of the polygon (by the closest point on a line, Theorem 7.13), the tangent line theorem (14.7) implies that each edge of the polygon is tangent to $c(O, r)$. Thus $P$ is tangential.

To prove uniqueness of incenter $O$ and the incircle $c(O, r)$, suppose that $c(O, r)$ and $c(O', r')$ are both incircles for $P$. Then the bisectors for each angle pass through both $O$ and $O'$, and since the angle bisectors are not collinear or parallel, this means that $O$ must equal $O'$. The radius is the distance from the center to any of the edges of the polygon, so $r$ must equal $r'$. Thus the incircles are unique.

**Theorem** (Theorem 14.33, altitude concurrence theorem, Exercise 14P). In any triangle, the lines containing the three altitudes are concurrent.

**Proof.** Let $\triangle ABC$ be a triangle, and copy it times: first find a point $B'$ so that $B$ and $B'$ are on opposite sides of $\overrightarrow{AC}$, and so that $\triangle AB'C \cong \triangle CBA$, and in particular so that $\angle CAB' \cong \angle ACB$, $\angle ACB' \cong \angle CAB$, and $\angle B' \cong \angle B$. Then repeat to find points $C'$ and $A'$, similarly, as in this figure:

Then the line $\overrightarrow{A'B'}$ is parallel to $\overrightarrow{AB}$, and so the altitude from $C$ to $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{A'B'}$, since it is perpendicular (by definition) to $\overrightarrow{AB}$. Indeed, this altitude is the perpendicular bisector of $\overrightarrow{A'B'}$. In fact, the altitudes of the original triangle are the perpendicular bisectors of $\overrightarrow{A'B'}, \overrightarrow{AC'},$ and $\overrightarrow{B'C'}$. Therefore by the perpendicular bisector concurrence theorem (Corollary 14.23), applied to the triangle $\triangle A'B'C'$, these three altitudes are concurrent.

**Theorem** (Exercise 14R). Suppose $\triangle ABC$ is a triangle.

1. The centroid, incenter, circumcenter, and orthocenter are the same point if and only if $\triangle ABC$ is equilateral.

2. If $\triangle ABC$ is isosceles but not equilateral, then the points all lie on the line going through the midpoint of the base and the opposite vertex.

**Proof.** (a) If $\triangle ABC$ is equilateral, then by the isosceles triangle altitude theorem (7.6), the altitude to each side is the median to that side and is also contained in the angle bisector. These are also contained in the perpendicular bisector of the corresponding side. So the centroid, incenter, circumcenter, and orthocenter are the same.

Conversely, suppose that these points are all the same for some triangle $\triangle ABC$; call the point of concurrency $O$. Consider the bisector of the angle $\angle A$ and the median from $A$ to $\overrightarrow{BC}$: these both go through $A$ and also through $O$, and therefore they are the same. Also, the perpendicular bisector of $\overrightarrow{BC}$ and the median both go through the midpoint $M$ of $\overrightarrow{BC}$ and $O$, so they coincide. Thus the segment $\overrightarrow{AM}$ bisects the angle $\angle A$, bisects the segment $\overrightarrow{BC}$, and is perpendicular to $\overrightarrow{BC}$; so by SAS, the triangles $\triangle ABM$ and $\triangle ACM$ are congruent, so $\overrightarrow{AB} \cong \overrightarrow{AC}$. Similarly, $\overrightarrow{AB} \cong \overrightarrow{BC}$, and so the triangle is equilateral.

(b) Suppose that $\triangle ABC$ is isosceles but not equilateral: suppose that $\overrightarrow{AB} \cong \overrightarrow{AC} \neq \overrightarrow{BC}$. Let $M$ be the midpoint of $\overrightarrow{BC}$; then $\overrightarrow{AM}$ is the median from $A$ to $\overrightarrow{BC}$, the angle bisector of $\angle A$, the perpendicular bisector of $\overrightarrow{BC}$, and the altitude from $A$, and thus all four points in question (the centroid, incenter, circumcenter, and orthocenter) lie on the line $\overrightarrow{AM}$. The centroid and the incenter both lie in the interior of the triangle, but the other two points need not. If the angle $\angle A$ is right, then $A$ itself is the orthocenter, while the circumcenter is $M$. If $\angle A$ is obtuse, then the orthocenter lies on the ray opposite to $\overrightarrow{AM}$, and the circumcenter lies on the ray opposite to $\overrightarrow{MA}$.

**Theorem** (Euler line theorem, Exercise 14S). The orthocenter, centroid, and circumcenter of any triangle are collinear.
Proof. Let $O$ be the circumcenter and $G$ the centroid. It suffices to assume that $O \neq G$: otherwise, we only have to show that at most two points are collinear, which is immediate. Let $H$ be the point such that $O \ast G \ast H$ and $GH = 2OG$. Let $A$ be a vertex of the triangle, and let $M$ be the midpoint of the opposite side. By the median concurrence theorem, we have $AG = 2GM$. The angles $\angle OGM$ and $\angle AGH$ are vertical angles, and hence congruent. Therefore the triangles $\triangle OGM$ and $\triangle HGA$ are similar, by SAS similarity. This means that $\angle OMG \cong \angle HAG$, so $\vec{OM}$ and $\vec{AH}$ are parallel, by the alternate interior angles theorem. Since $\vec{OM}$ is perpendicular to $\vec{BC}$, so therefore is $\vec{AH}$. Thus the portion of $\vec{AH}$ from $A$ to the line $\vec{BC}$ is the altitude of the triangle from $A$ to $\vec{BC}$. This means that $H$ lies on this altitude.

Repeating this argument with the other vertices and midpoints shows that $H$ lies on all three altitudes, as desired. \qed