Geometric Categorification
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Finite Dimensional $\mathfrak{sl}_2$ modules

An action of $U(\mathfrak{sl}_2)$ on a finite dimensional vector space $V$ consists of

- a weight space decomposition $V = \bigoplus_{\lambda \in \mathbb{Z}} V(\lambda)$,
- linear maps

\[
e(\lambda) : V(\lambda - 1) \to V(\lambda + 1)
\]
\[
f(\lambda) : V(\lambda + 1) \to V(\lambda - 1)
\]

for each $\lambda$. These satisfy

\[
e(\lambda - 1)f(\lambda - 1) = \lambda \text{Id}_{V(\lambda)} + f(\lambda + 1)e(\lambda + 1).
\]

The element $s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in SL_2$ acts on $V$, giving isomorphisms

\[
s : V(-\lambda) \to V(\lambda).
\]
\(\mathfrak{sl}_2\) Categorification

Frameworks for \(\mathfrak{sl}_2\) categorification have been proposed by Chuang-Rouquier, Lauda, Khovanov-Lauda, and Rouquier. They define (a collection of closely related) categories \(U_2(\mathfrak{sl}_2)\) which categorify the enveloping algebra \(U(\mathfrak{sl}_2)\).

Our Goal: Construct geometric representations of \(U_2(\mathfrak{sl}_2)\).

Ordinary representations of \(\mathfrak{sl}_2\) have been constructed geometrically by Lusztig, Ginzburg, Nakajima, and others. So it is natural to look for geometric examples of categorified representations.
Geometric $\mathfrak{sl}_2$ Categorification

- Weight spaces get replaced by varieties:

$$V(\lambda) \mapsto Y(\lambda).$$

- Linear transformations get replaced by Fourier-Mukai Kernels

$$e \mapsto \mathcal{E}, \quad f \mapsto \mathcal{F},$$

$$\mathcal{E}(\lambda), \mathcal{F}(\lambda) \in D^b(Y(\lambda - 1) \times Y(\lambda + 1)).$$

These kernels are required to satisfy $\mathfrak{sl}_2$ relations, but only at the level of cohomology of complexes.

- We require the existence of deformations

$$\tilde{Y}(\lambda) \longrightarrow \mathbb{A}^1$$

with some special properties.
One main virtue of geometric $\mathfrak{sl}_2$ categorifications is that they give rise to categorified representations.

**Theorem (Cautis-Kamnitzer-L)**

A geometric $\mathfrak{sl}_2$ categorification induces a representation of $U_2(\mathfrak{sl}_2)$ on

$$\bigoplus_{\lambda} D^b(Y(\lambda)).$$

The functors $E$ and $F$ are induced by the kernels $E$ and $F$, while the natural transformations ($X$ and $T$, $y$ and $\psi$, or dots and crosses) are constructed using the deformations $\tilde{Y}(\lambda)$. 
A Basic Example

Fix $N \in \mathbb{N}$. For $0 \leq k \leq N$, set

$$Y(2k - N) = T^* \text{Gr}(k, N) \cong$$

$$\{(x, V) : x \in M_N(\mathbb{C}), 0 \subset V \subset \mathbb{C}^N, \dim(V) = k \text{ and } \mathbb{C}^N \xrightarrow{x} V \xrightarrow{x} 0\}$$

There are tautological bundles

- $V$,
- $\mathbb{C}^N / V$

on $Y(\lambda)$.

There are also natural deformations $\tilde{Y}(\lambda)$, given by varying the action of $x$ on $V$ and $\mathbb{C}^N / V$. 
Hecke Correspondences

For $r \geq 0$, define $W^r(\lambda) \subset Y(\lambda - r) \times Y(\lambda + r)$ by

$W^r(\lambda) := \{(x, V, V') : 0 \subset V \subset V' \subset \mathbb{C}^N; \mathbb{C}^N \xrightarrow{x} V \text{ and } V' \xrightarrow{x} 0\}$.

Projections:

$\pi_1 : (x, V, V') \mapsto (x, V), \quad \pi_2 : (x, V, V') \mapsto (x, V')$.

Tautological bundles on $W^r(\lambda)$:

- $V := \pi_1^*(V)$
- $V' := \pi_2^*(V)$

Inclusions:

$0 \subset V \subset V' \subset \mathbb{C}^N \cong O_{W^r(\lambda)}$. 
Kernels $\mathcal{E}(\lambda), \mathcal{F}(\lambda)$ for the Basic Example

Define the kernel $\mathcal{E}^{(r)}(\lambda) \in D(Y(\lambda - r) \times Y(\lambda + r))$ by

$$\mathcal{E}^{(r)}(\lambda) := \mathcal{O}_{W^r(\lambda)} \otimes \det(V' / V)^{-\lambda}.$$ 

Similarly, $\mathcal{F}^{(r)}(\lambda) \in D(Y(\lambda + r) \times Y(\lambda - r))$ is defined by

$$\mathcal{F}^{(r)}(\lambda) := \mathcal{O}_{W^r(\lambda)} \otimes \det(C^N / V')^{-r} \det(V)^{r}.$$
Functors $E$ and $F$ from kernels $\mathcal{E}$ and $\mathcal{F}$.

A kernel $\mathcal{A} \in D(Y(\lambda) \times Y(\lambda'))$ induces a functor

$$\Phi_{\mathcal{A}} : D(Y(\lambda)) \longrightarrow D(Y(\lambda'))$$

given by

$$y \mapsto \pi_2^* (\pi_1^*(y) \otimes \mathcal{A}).$$

Thus the kernels $\mathcal{E}^{(r)}(\lambda), \mathcal{F}^{(r)}(\lambda)$ give rise to functors

$$E^{(r)}(\lambda) := \Phi_{\mathcal{E}^{(r)}(\lambda)} : D(Y(\lambda - r)) \longrightarrow D(Y(\lambda + r))$$

$$F^{(r)}(\lambda) := \Phi_{\mathcal{F}^{(r)}(\lambda)} : D(Y(\lambda + r)) \longrightarrow D(Y(\lambda - r)).$$
Relations satisfied by the Es and Fs

- The E’s and F’s are biadjoint up to shifts.
- E’s (and hence F’s) compose as
  \[ E(\lambda + r) \circ E(r)(\lambda - 1)) \cong E^{(r+1)}(\lambda) \otimes IH^*(\mathbb{P}^r) \]
  via an explicit isomorphism built from the data.
- If \( \lambda \leq 0 \) then
  \[ F(\lambda + 1) \circ E(\lambda + 1) \cong E(\lambda - 1) \circ F(\lambda - 1) \oplus \text{Id} \otimes IH^*(\mathbb{P}^{-\lambda-1}) \]
  via an explicit isomorphism built from the data. (Similarly for \( \lambda \geq 0 \).)
Application: Equivalences

Given any geometric $\mathfrak{sl}_2$ categorification, we build a complex of functors

$$\Theta_* : \mathcal{D}(-\lambda) \rightarrow \mathcal{D}(\lambda).$$

Fix $\lambda \geq 0$. For $r = 0, \ldots, N - \lambda$, set

$$\Theta_r = E^{(\lambda+r)}(-r)F^{(r)}(-\lambda - r)[-r].$$

The differential $\Theta_r \rightarrow \Theta_{r-1}$ is constructed using units and counits of adjunctions between $E^{(k)}$ and $F^{(k)}$. The complex $\Theta_*$ categorifies the action of the reflection element $s \in SL_2$, and was considered first by Chuang-Rouquier.
Equivalences

**Theorem (Cautis-Kamnitzer-L)**

*The complex $\Theta_*$ is an equivalence between the opposite $\mathfrak{sl}_2$ weight space categories $\mathcal{D}(-\lambda)$ and $\mathcal{D}(\lambda)$.

Applied to the basic example:

**Corollary**

*The complex $\Theta_*$ gives an equivalence

$$\Theta_* : D(T^*(Gr(k, N))) \to D(T^*(Gr(N - k, N))).$$

This answers questions posed by Kawamata and Namikawa.
From $\mathfrak{sl}_2$ to $\mathfrak{g}$

There are analogous definitions of geometric categorification when $\mathfrak{g}$ is a Kac-Moody Lie algebra.

- Examples: Nakajima Quiver Varieties.
- Conjecturally, geometric categorifications induce representations of $U_2(\mathfrak{g})$.
- Braid group actions: For each root $\mathfrak{sl}_2$ inside $\mathfrak{g}$, we have an equivalence $\Theta_{i*}$ given by the Chuang-Rouquier complex.

**Theorem (Cautis-Kamnitzer)**

*The equivalences $\{\Theta_{i*}\}$ coming from a geometric $\mathfrak{g}$ categorification define an action of the braid group.*