Irreducible representations of Khovanov-Lauda-Rouquier algebras of finite type

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Lie theoretic data

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$$A_2^{(1)} : \quad \begin{array}{c}
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- \( \mathcal{W} := \bigsqcup_{d \geq 0} I^d \) (words in the alphabet \( I \)).
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- $W := \bigsqcup_{d \geq 0} I^d$ (words in the alphabet $I$);
- for $\alpha \in Q_+$, define words of weight $\alpha$:

$$W^\alpha := \{i = (i_1, \ldots, i_d) \in W \mid \alpha_{i_1} + \cdots + \alpha_{i_d} = \alpha\}.$$
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Each \( R_\alpha \) (= a block of \( R_d \)) is a unital \( F \)-algebra generated by

\[ \{ e(i) \mid i \in W^\alpha \} \cup \{ y_1, \ldots, y_d \} \cup \{ \psi_1, \ldots, \psi_{d-1} \} \]

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and some relations. For example, the relations say that \( e(i) \)'s are mutually orthogonal idempotents which sum to 1, that \( e \)'s and \( y \)'s commute, there is an important relation

\[ e(i)\psi_r = \psi_r e((r, r + 1) \cdot i), \]
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  e(i) & \text{if } i_r = i_{r+1}, \\
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$$\psi_r^2 e(i) = Q_{i_r,i_{r+1}}(y_r,y_{r+1})e(i).$$

for certain explicit polynomial $Q_{i_r,i_{r+1}}$ depending on $c_{i_r,i_{r+1}}$ and orientation (i.e. on how $i_r$ and $i_{r+1}$ are connected in the Dynkin diagram).
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A ($\mathbb{Z}$-)grading on $R_\alpha$ is defined by prescribing explicit degrees to the generators $e(i)$, $y_r e(i)$, and $\psi_r e(i)$. 
Motivation for KLR algebras

**Motivation 1:** Khovanov-Lauda and Rouquier used $R_d$ to categorify quantum groups.

More precisely, the Khovanov-Lauda theorem says that the category of finitely generated projective graded modules over the algebras $R_d$ for all $d \in \mathbb{Z}_{\geq 0}$ categorify the negative part $f$ of the quantum group corresponding to the Cartan matrix $C$.

This leads to a definition of 2-Kac-Moody algebras and further categorical generalizations of Kac-Moody algebras, quantum groups and modules over them.

**Motivation 2:** Brundan-K.’08 constructed explicit isomorphisms between the usual cyclotomic Hecke algebras $H_\Lambda d$ and the corresponding cyclotomic quotients $R_\Lambda d$ of $R_d$ for $C$ “of type A”:

$$H_\Lambda d \cong R_\Lambda d (C)$$

This sheds some new light on the classical representation theory of Hecke algebras and symmetric groups, for example allowing us to grade the corresponding irreducible modules and Specht modules, and so on.
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This sheds some new light on the classical representation theory of Hecke algebras and symmetric groups, for example allowing us to grade the corresponding irreducible modules and Specht modules, study *graded decomposition numbers*, and so on.
The cyclotomic Hecke algebra $H^\Lambda_d$ depends on the parameter $q \in F^\times$. 

Let $e$ be the smallest positive integer such that $1 + q + \cdots + q^{e-1} = 0$, set $e := 0$ if no such integer exists.

E.g. if $q = 1$, then $e = \text{char } F$.

Theorem (Brundan-K.'08) Let the Cartan matrix $C$ be of type $C := \begin{cases} A_\infty & \text{if } e = 0, \\ A^{(1)}_{e-1} & \text{if } e > 0. \end{cases}$ Then $H^\Lambda_d \cong R^\Lambda_d$. So representation theory of $R^\Lambda_d(A_\infty)$ is equivalent to representation theory of affine Hecke algebras in characteristic zero (or with generic parameter), while representation theory of $R^\Lambda_d(A^{(1)}_{e-1})$ is equivalent to modular representation theory in "characteristic $e$."
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One should think about representation theory of $R_d(C)$ as representation theory of symmetric group $S_d$ (and more generally the corresponding affine Hecke algebra $H_d$) “in characteristic $C$”. 

Goal: classify the irreducible modules over $R_d(C)$ for the Cartan matrix $C$ of finite type. Assume from now that $C$ is of finite type ($A_\infty, B_\infty, C_\infty$ and $D_\infty$ are also allowed). (It was noticed by Khovanov and Lauda that irreducible $R_d$-modules are always finite dimensional.)
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For example representation theory of $S_d = H_d^\Lambda_0$ “in characteristic $E_8$” is representation theory of $R_d^\Lambda_0(E_8)$ which is contained in representation theory of $R_d(E_8)$ just like representation theory of $S_d$ is contained in representation theory of the corresponding (degenerate) affine Hecke algebra $H_d$. 

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(It was noticed by Khovanov and Lauda that irreducible $R_d$-modules are always finite dimensional.)
Let $V$ be a finite dimensional $R_\alpha$-module and $i \in W^\alpha$. 

Word theory (think *weight theory*)!
Let $V$ be a finite dimensional $R_\alpha$-module and $i \in W^\alpha$. We refer to $V_i := e(i)V$ as the $i$-word space of $V$. 

To be able to speak of a highest word, pick any total order on $I$. This induces lexicographic order "$\leq$" on the words.

Theorem: The isomorphism class of an irreducible $R_\alpha$-module $L$ is determined by the highest word of $L$. 

Notation: if $i$ is the highest word of an irreducible $R_\alpha$-module $L$, we denote $L$ by $L(i)$.
Let $V$ be a finite dimensional $R_\alpha$-module and $i \in W^\alpha$. We refer to $V_i := e(i)V$ as the $i$-word space of $V$. We have word space decomposition:

$$V = \bigoplus_{i \in W^\alpha} V_i.$$
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**Notation:** if $i$ is the highest word of an irreducible $R_\alpha$-module $L$, we denote $L$ by $L(i)$. 
A word \( i \in W^\alpha \) is called \textit{dominant} if and only if it occurs as a highest word of some (irreducible) \( R_\alpha \)-module. The set of all dominant words in \( W^\alpha \) is denoted by \( W^\alpha_+ \).
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So the theorem above can be interpreted as the statement that

\[ \{ L(i) \mid i \in \mathbf{W}^\alpha_+ \} \]

is a complete and irredundant set of irreducible \( R^\alpha \)-modules.
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The goal now is to describe the set of dominant words and to construct the simple modules as heads of certain standard modules.
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**Classical fact:** every word $i$ has a unique factorization

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such that $i^{(1)} \geq i^{(2)} \geq \cdots \geq i^{(k)}$ are Lyndon words. This is called the *canonical factorization* of $i$. 
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**Theorem**

Let $i \in \mathcal{W}_\alpha$ and

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be the canonical factorization of $i$. Then $i$ is dominant if and only if each $i^{(k)}$ is dominant.
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Let $i \in W^\alpha$ and

$$i = i^{(1)}i^{(2)} \ldots i^{(k)}$$

be the canonical factorization of $i$. Then $i$ is dominant if and only if each $i^{(k)}$ is dominant.

Thus we are reduced to describing only dominant Lyndon words, which we call *minuscule* words.
The following follows from Lalonde-Ram’95 and Leclerc’04:

**Theorem**

(i) There is a minuscule word in $W_{\alpha}$ if and only if $\alpha \in \Phi^+$, in which case there is exactly one minuscule word in $W_{\alpha}$. Denote this word by $i_{\alpha}$.

Thus $\Phi^+ \rightarrow \{\text{minuscule words}\}$, $\beta \mapsto i_{\beta}$ is a bijection between the set of positive roots and the set of all minuscule words.

(ii) Let $\beta \in \Phi^+$. Then $i_{\beta}$ is the smallest element among $W_{\beta}^+$. 

(iii) Let $\beta \in \Phi^+$ and $C(\beta) = \{(\beta_1, \beta_2) \in \Phi^+ \times \Phi^+ | \beta_1 + \beta_2 = \beta, i_{\beta_1} < i_{\beta_2}\}$.

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The idea now is that minuscule modules should be easy to construct explicitly and then other irreducible modules could be constructed out of them using induction (hence the competing term “cuspidal”).
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E.g. if you choose one of the two natural orderings in type $A_n$:

$$1 < 2 < \cdots < n \quad \text{or} \quad 1 > 2 > \cdots > n$$

then the minuscule modules are just one-dimensional and they correspond to Zelevinsky segments.
Originally, we were able to construct the miniscule modules in all types other than $E_8$ and $F_4$ (for some specific natural choice of the ordering on $I$). In type $E_8$ we could construct them for all but 12 positive positive roots.

Fortunately, this issue has been resolved recently by Hill, Melvin, and Mondragon, who were able to construct all cuspidal modules for certain natural orderings on $I$. A special case of their result which improves our work in type $E_8$ is

**Theorem (Hill-Melvin-Mondragon'09)**

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Let $\alpha, \beta \in Q_+$. There is an obvious (non-unital) algebra embedding of $R_\alpha \otimes R_\beta$ into the $R_{\alpha+\beta}$ mapping $e(i) \otimes e(j)$ to $e(ij)$. 
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$$\text{Ind}_{\alpha,\beta}^{\alpha+\beta} := R_{\alpha+\beta} e_{\alpha,\beta} \otimes R_\alpha \otimes R_\beta : R_\alpha \otimes R_\beta\text{-Mod} \to R_{\alpha+\beta}\text{-Mod}.$$
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For $\alpha, \beta \in Q_+$, $M \in \text{Rep}(R_\alpha)$ and $N \in \text{Rep}(R_\beta)$, we denote

$$M \circ N := \text{Ind}_{\alpha, \beta}^{\alpha+\beta} (M \boxtimes N).$$
Constructing all irreducible modules

Let $i \in W^\alpha_+$. We want to construct $L(i)$. 

Write the canonical factorization of $i$:

$$i = i^{(1)} i^{(2)} \cdots i^{(k)},$$

i.e. $i^{(1)} \geq i^{(2)} \geq \cdots \geq i^{(k)}$ are minuscule words.

Define the standard module of highest weight $i$:

$$\Delta(i) := L(i^{(1)}) \circ \cdots \circ L(i^{(k)}).$$

Theorem

Let $\alpha \in Q^+, i \in W^\alpha_+$, and $\Delta(i)$ be the standard $R^\alpha$-module. Then:

(i) The highest word of $\Delta(i)$ is $i$.

(ii) $\Delta(i)$ has an irreducible head $L(i)$.

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*Let $C$ be a Cartan matrix of finite type. Then the formal characters of irreducible $R_d(C)$-modules are independent of the characteristic of the ground field $F$.***

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A different classification of irreducible modules over KLR algebras was obtained by Lauda and Vazirani.
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