CONSTRUCTIONS FOR INFINITESIMAL GROUP SCHEMES

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Abstract. Let G be an infinitesimal group scheme over a field k of characteristic $p > 0$. We introduce the universal p-nilpotent operator $\Theta_G \in \mathrm{Hom}_k(k[G], k[V(G)])$, where $V(G)$ is the scheme which represents 1-parameter subgroups of G. This operator Θ_G applied to M encodes the local Jordan type of M, and leads to computational insights into the representation theory of G. For certain kG-modules M (including those of constant Jordan type), we employ Θ_G to associate various algebraic vector bundles on $\mathbb{P}(G)$, the projectivization of $V(G)$. These vector bundles not only distinguish certain representations with the same local Jordan type, but also provide a method of constructing algebraic vector bundles on $\mathbb{P}(G)$.

Contents

0. Introduction

In recent years, techniques have been developed by the authors ([12], [14]) to investigate representations of an arbitrary finite group scheme over a field k of characteristic $p > 0$ in a manner which extends earlier work for elementary abelian p-groups and p-restricted finite dimensional Lie algebras ([4], [10]). One general class of such finite group schemes is the class of infinitesimal group schemes, which includes Frobenius kernels of algebraic groups over k such as those corresponding to p-restricted Lie algebras (arising as infinitesimal group schemes of height 1). In this paper, we introduce and study several new families of invariants, both geometric and numerical, for representations of an infinitesimal group scheme. Although such representations are less familiar than representations of finite groups, their relevance to other representation theories is reflected in the fact that the representation theories of the family of all infinitesimal kernels $G = \mathfrak{G}_{(r)}$ of a connected

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reductive algebraic group $\mathfrak{G}$ is essentially equivalent to the rational representation theory of $\mathfrak{G}$.

In [22], [23], the foundations of a theory of "support varieties" were established for an infinitesimal group scheme G based upon the 1-parameter subgroups of G. In contrast to the theory for finite groups, one encounters cohomological support varieties of considerable geometric complexity and one is challenged by the difficulty of computing explicit examples. We develop techniques to study representations of infinitesimal group schemes which we then apply to four fundamental, yet concrete, classes of examples. Namely, we consider

i.) p-restricted Lie algebras, $\mathfrak{g}$,
ii.) infinitesimal additive group schemes, $\mathbb{G}_{a(r)}$,
iii.) infinitesimal general linear groups, $\mathrm{GL}_{n(r)}$,
iv.) the height 2, infinitesimal special linear group, $\mathrm{SL}_{2(2)}$.

While we obtain various general results for representations of arbitrary infinitesimal group schemes, we also seek to consistently make those results concrete by applying them to our examples.

Recall that a finite group scheme G has a finite dimensional commutative coordinate algebra $k[G]$ whose k-linear dual kG is a cocommutative Hopf algebra, the group algebra of G. In particular, a representation of G (over k) is precisely a kG-module. A finite group scheme is said to be infinitesimal if its coordinate algebra $k[G]$ is a local ring. Thus, within finite group schemes, infinitesimal group schemes are at the opposite end of the spectrum from finite groups (whose coordinate algebras are etale over the ground field). Infinitesimal 1-parameter subgroups as considered in [22], [23] determine natural representatives of equivalence classes of π-points as defined in [14]. The representability of the functor of 1-parameter subgroups associated to an infinitesimal group scheme provides a universal p-nilpotent operator which we exploit. The special features of infinitesimal group schemes enable us to provide constructions associated to their representations which are not available for finite groups and other types of finite group schemes.

Our general constructions yield algebraic vector bundles on the projectivization of the cohomological support variety of an arbitrary infinitesimal group scheme G associated to special representations of G, those of constant j-type. Since cohomological support varieties are singular, yet familar affine varieties (for example, the variety for $\mathrm{GL}_{n(r)}$ is the variety of r-tuples of pairwise commuting p-nilpotent matrices), we anticipate that this explicit construction of vector bundles will provide a useful technique in the study of certain projective varieties whose K-theoretic invariants have been inaccessible. The reader might find it instructive to contrast our construction of vector bundles on $\mathbb{P}(G)$ from representations of G with the Borel-Weil construction of representations from explicit line bundles on flag varieties.

In Section 1, we recall some of the highlights from [22], [23] concerning the cohomology and theory of supports of finite dimensional kG-modules for an infinitesimal group scheme G. A key result summarized in Theorem 1.15 is the close relationship between the spectrum $\operatorname{Spec} \mathrm{H}^\bullet(G,k)$ of the cohomology of G and the scheme $V(G)$ representing 1-parameter subgroups of an infinitesimal group scheme G. Here, and throughout this paper, we illustrate these general results with our four representative examples.

In the second section, we define the universal p-nilpotent operator

$$\Theta_G \in \operatorname{Hom}_k(k[G], k[V(G)]) \simeq k[V(G)] \otimes kG,$$

for an infinitesimal group scheme G. For any finite dimensional kG-module M, Θ_G determines a p-nilpotent endomorphism of the free $k[V(G)]$-module $k[V(G)] \otimes M$. We establish in Proposition 2.8 that Θ_G is homogeneous of degree p^{r-1}, where $k[V(G)]$ is equipped with its natural grading and $V(G)$ is identified with the scheme of 1-parameter subgroups $V_r(G)$ of height r for any choice of r with $r \geq \mathrm{ht}(\mathrm{G})$.

In the third section, we verify that specializations θ_v of Θ_G at points $v \in V(G)$ when applied to a finite dimensional kG-module M determine the local Jordan type of M. Theorem 3.6 can be viewed as providing an algorithm for obtaining the local Jordan type in terms of the representation $G \to \mathrm{GL}_N$ defining the kG-module M. We utilize Θ_G and its specializations to establish constraints for a kG-module M to be of constant rank (and thus of constant Jordan type). We also produce a relationship between the local Jordan type of a module and its Frobenius twists.

We envision that some of our constructions for infinitesimal group schemes may lead to analogues for a general finite group scheme. With this in mind, we provide in the fourth section a dictionary between 1-parameter subgroups for infinitesimal group schemes and π-points for general finite group schemes. In particular, we recall from [14] the existence of a bijective morphism of projective schemes $\mathbb{P}(\mathrm{G}) \to \Pi(G)$, where $\mathbb{P}(\mathrm{G}) = \operatorname{Proj} V(G)$. Here, $\Pi(G)$ is the scheme of equivalence classes of π-points introduced in [14]. Given a finite dimensional kG-module M, we consider the projectivization of the operator Θ_G,

$$\widetilde{\Theta}_G : \mathcal{O}_{\mathbb{P}(\mathrm{G})} \otimes M \to \mathcal{O}_{\mathbb{P}(\mathrm{G})}(p^{r-1}) \otimes M,$$

a p-nilpotent operator on the free, coherent sheaf $\mathcal{O}_{\mathbb{P}(\mathrm{G})} \otimes M$ on $\mathbb{P}(\mathrm{G})$. We verify in Proposition 4.7 that $\widetilde{\Theta}_G$ determines via base change the local Jordan type of a kG-module M at any 1-parameter subgroup $\mu_v : \mathbb{G}_{a(r),k(v)} \to G_{k(v)}$.

Theorem 4.12 shows that the condition that M be of constant j-rank is equivalent to the condition that the coherent sheaf $\operatorname{Im} \widetilde{\Theta}_G^j$ be locally free. In the fifth section, we initiate an investigation of the algebraic vector bundles $\operatorname{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$, $\operatorname{Im}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$ on $\mathbb{P}(\mathrm{G})$ associated to kG-modules of constant Jordan type and more generally of constant j-rank. We give examples of such kG-modules in each of our four representative examples. As we see, taking kernels of powers of the universal p-nilpotent power leads to vector bundles, as does taking kernels modulo images (as inspired by a construction of M. Duflo and V. Serganova for Lie superalgebras in [8]). For example, Proposition 5.15 discusses a construction for any finite dimensional kG-module M which results in an algebraic line bundle if and only if M is an endotrivial module.

Finally, in the last section, we provide numerous explicit examples. These include the infinitesimal group scheme $G = \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$, which has the same representation theory as an elementary abelian p-group $\mathbb{Z}/p \times \mathbb{Z}/p$, as well as the first Frobenius kernel of the reductive group SL_2. One intriguing comparison which we investigate in particularly simple examples is the relationship between the Grothendieck group of projective kG-modules and the Grothendieck group of algebraic vector bundles on $\mathbb{P}(G)$. Combined with our explicit calculations, Proposition 6.7 can be viewed both as a means to distinguish certain non-isomorphic projective kG-modules and as a means of constructing non-isomorphic algebraic vector bundles on $\mathbb{P}(\mathrm{G})$. Such calculation lead to lower bounds on the rank of $K_0(\mathbb{P}(G))$ for certain infinitesimal group schemes G. We conclude with other specific examples of kG-modules which lead to interesting bundles, "zig-zag modules" and syzygies $\Omega^n(k), n \in \mathbb{Z}$, of the trivial kG-module k.

Throughout, k will denote an arbitrary field of characteristic $p > 0$. Unless explicit mention is made to the contrary, G will denote an infinitesimal group scheme over k. If M is a kG-module and K/k is a field extension, then we denote by M_K the KG-module obtained by base extension.

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1. Infinitesimal group schemes

The purpose of this first section is to summarize the important role played by (infinitesimal) 1-parameter subgroups of an infinitesimal group scheme as presented in [22]. The four representative examples of Example 1.4, ($\underline{\mathfrak{g}}$, $\mathbb{G}_{a(r)}$, $\mathrm{GL}_{n(r)}$, $\mathrm{SL}_{2(2)}$), and their associated schemes of 1-parameter subgroups discussed in Example 1.11 will serve as explicit models to which we will frequently return.

Definition 1.1. A finite group scheme G over k is a group scheme over k whose coordinate algebra $k[G]$ is finite dimensional over k.

Equivalently, G is a functor from commutative k-algebras to groups, $R \mapsto G(R)$, represented by a finite dimensional commutative k-algebra, the coordinate algebra $k[G]$ of G.

Associated to G, we have its group algebra $kG = \mathrm{Hom}_k(k[G], k)$; more generally, for any commutative k-algebra R, we have the R-group algebra $RG = \mathrm{Hom}_k(k[G], R)$.

Observe that the R-group algebra of G consists of all k-linear homomorphisms, whereas $G(R) = \mathrm{Hom}_{k-alg}(k[G], R)$ is the subgroup of RG consisting of k-algebra homomorphisms.

Definition 1.2. Let G be a finite group scheme over k and M a k-vector space. Then a kG-module structure on M is given by one of the following equivalent sets of data (see, for example, [19]):

- The structure $M \to k[G] \otimes M$ of a $k[G]$-comodule on M.
- The structure $kG \otimes M \to M$ of a kG-module on M.
- A functorial (with respect to R) group action $G(R) \times (R \otimes M) \to (R \otimes M)$.

For most of this paper we shall restrict our consideration to infinitesimal group schemes, a special class of finite group schemes which we now define.

Definition 1.3. An infinitesimal group scheme G (over k) of height $\leq r$ is a finite group scheme whose coordinate algebra $k[G]$ is a local algebra with maximal ideal $\mathfrak{m}$ such that $x^{p^r} = 0$ for all $x \in \mathfrak{m}$.

Example 1.4. We shall frequently consider the following four examples.

(1) A finite dimensional p-restricted Lie algebra $\mathfrak{g}$ corresponds naturally with a height 1 infinitesimal group scheme which we denote $\underline{\mathfrak{g}}$ ([19, I.8.5]). The group algebra of $\underline{\mathfrak{g}}$ is the restricted enveloping algebra $\mathfrak{u}(\mathfrak{g})$ of $\mathfrak{g}$. If $\mathfrak{g}$ is the Lie algebra of

a group scheme $\mathfrak{G}$, then the coordinate algebra of $\underline{\mathfrak{g}}$ is given by $k[\mathfrak{G}]/(x^p, x \in \mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal of $k[\mathfrak{G}]$ at the identity of $\mathfrak{G}$.

(2) Let $\mathbb{G}_a$ denote the additive group, so that $k[\mathbb{G}_a] = k[t]$ with coproduct defined by $\nabla(t) = t \otimes 1 + 1 \otimes t$. As a functor, $\mathbb{G}_a : (comm\ k-alg) \to (grps)$ sends an algebra R to its underlying abelian group. For any $r \geq 1$, we consider the r$^{\text{th}}$ Frobenius kernel of $\mathbb{G}_a$,

$$\mathbb{G}_{a(r)} \ \equiv \mathrm{Ker}\{F^r : \mathbb{G}_a \to \mathbb{G}_a\}.$$

Here $F : \mathbb{G}_a \to \mathbb{G}_a$ is the (geometric) Frobenius specified by its map on coordinate algebras $k[t] \to k[t]$ given as the k-linear map sending t to t^p. The coordinate algebra of $\mathbb{G}_{a(r)}$ is given by $k[\mathbb{G}_{a(r)}] = k[t]/t^{p^r}$, whereas the group algebra of $\mathbb{G}_{a(r)}$ is given by

$$k\mathbb{G}_{a(r)} \simeq k[\mathbb{G}_{a(r)}]^{\#} \simeq k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p), \tag{1.4.1}$$

where u_i is a linear dual to t^{p^i}, $0 \leq i \leq r-1$.

(3) Let GL_n denote the general linear group, the representable functor sending a commutative algebra R to the group $\mathrm{GL}_n(R)$. For any $r \geq 1$, we consider the r$^{\text{th}}$ Frobenius kernel of GL_n,

$$\mathrm{GL}_{n(r)} \ \equiv \mathrm{Ker}\{F^r : \mathrm{GL}_n \to \mathrm{GL}_n\},$$

where the geometric Frobenius

$$F : \mathrm{GL}_n(R) \to \mathrm{GL}_n(R)$$

is defined by raising each matrix entry to the p$^{\text{th}}$ power. The coordinate algebra of $\mathrm{GL}_{n(r)}$ is given by

$$k[\mathrm{GL}_{n(r)}] \ = \ \frac{k[X_{ij}]}{(X_{ij}^{p^r} - \delta_{ij})}_{1 \leq i,j \leq n},$$

whereas the group algebra of $\mathrm{GL}_{n(r)}$ is given as

$$k\,\mathrm{GL}_{n(r)} \ = \ \mathrm{Hom}_k(k[\mathrm{GL}_{n(r)}], k),$$

the k-space of linear functionals $k[\mathrm{GL}_{n(r)}]$ to k. The coproduct

$$\nabla : k[\mathrm{GL}_{n(r)}] \ \to \ k[\mathrm{GL}_{n(r)}] \otimes k[\mathrm{GL}_{n(r)}]$$

is given by sending X_{ij} to $\sum_k X_{ik} \otimes X_{kj}$.

(4) The height 2 infinitesimal group scheme $\mathrm{SL}_{2(2)}$ is essentially a special case of $\mathrm{GL}_{n(r)}$. This is once again defined as the kernel of an iterate of Frobenius

$$\mathrm{SL}_{2(2)} \ \equiv \mathrm{Ker}\{F^2 : \mathrm{SL}_2 \to \mathrm{SL}_2\}.$$

The coordinate algebra of $\mathrm{SL}_{2(2)}$ is given by

$$k[\mathrm{SL}_{2(2)}] = \frac{k[X_{11}, X_{12}, X_{21}, X_{22}]}{(X_{11}X_{22} - X_{12}X_{21} - 1, X_{ij}^{p^2} - \delta_{ij})}$$

whereas the group algebra of $\mathrm{SL}_{2(2)}$ is given as

$$k\,\mathrm{SL}_{2(2)} = k\langle e, f, h, e^{(p)}, f^{(p)}, h^{(p)}\rangle / \langle relations\rangle$$

with $e,\ f,\ h,\ e^{(p)},\ f^{(p)},\ h^{(p)}$ the dual basis vectors to $X_{12},\ X_{21},\ X_{11} - 1,\ X_{12}^p,\ X_{21}^p,\ (X_{11}-1)^p$ respectively.

Definition 1.5. A (infinitesimal) 1-parameter subgroup of height r of an affine group scheme G_R over a commutative k-algebra R is a homomorphism of R-group schemes $\mathbb{G}_{a(r),R} \to G_R$.

We recall the description of height r 1-parameter subgroups of GL_n given in [22].

Proposition 1.6. [22, 1.2] *If $G = \mathrm{GL}_n$ and if R is a commutative k-algebra, then a 1-parameter subgroup of $\mathrm{GL}_{n,R}$ of height r, $f : \mathbb{G}_{a(r),R} \to \mathrm{GL}_{n,R}$, is naturally (with respect to R) equivalent to a comodule map*

$$\Delta_f : R^n \to R[t]/t^{p^r} \otimes_R R^n, \quad \Delta_f(v) = \sum_{j=0}^{p^{r-1}} t^j \otimes \beta_j(v), \quad \beta_j \in M_n(R)$$

satisfying the constraints of being counital and coassociative. This in turn is equivalent to specifying an r-tuple of matrices $\alpha_0 = \beta_0, \alpha_1 = \beta_p, \ldots, \alpha_{r-1} = \beta_{p^{r-1}}$ in $M_n(R)$ such that each α_i has p^{th} power 0 and such that the α_i's pairwise commute. The other coefficient matrices β_j are given by the formula

$$\beta_j = \frac{\alpha_0^{j_0} \cdots \alpha_{r-1}^{j_{r-1}}}{(j_0)! \cdots (j_{r-1})!} \in M_n(R), \quad j = \sum_{i=0}^{r-1} j_i p^i \text{ with } 0 \le j_i < p. \tag{1.6.1}$$

As shown in [22], Proposition 1.6 implies the following representability of the functor of 1-parameter subgroups of height r.

Theorem 1.7. [22, 1.5] *For any affine group scheme G, the functor from commutative k-algebras to sets*

$$R \ \mapsto \ \mathrm{Hom}_{\mathrm{grp\,sch}}(\mathbb{G}_{a(r),R}, G_R)$$

is representable by an affine scheme $V_r(G) = \operatorname{Spec} k[V_r(G)]$. Namely, this functor is naturally isomorphic to the functor

$$R \ \mapsto \ \mathrm{Hom}_{\mathrm{k-alg}}(k[V_r(G)], R).$$

By varying r, we can associate a family of affine schemes to an affine group scheme G. In the following remark we make explicit the relationship between various $V_r(G)$ for the same G and varying r's.

Remark 1.8. For $r > s \ge 1$, let $p_{r,s} : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(s)}$ be the canonical projection given by the natural embedding of the coordinate algebras

$$p_{r,s}^* : k[\mathbb{G}_{a(s)}] = k[t]/t^{p^s} \xrightarrow{t \to t^{p^{r-s}}} k[t]/t^{p^r} = k[\mathbb{G}_{a(r)}].$$

The corresponding map on group algebras

$$k\mathbb{G}_{a(r)} \simeq k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p) \xrightarrow{p_{r,s,*}} k\mathbb{G}_{a(s)} \simeq k[v_0, \ldots, v_{s-1}]/(v_0^p, \ldots, v_{s-1}^p)$$

sends $\{u_0, \ldots, u_{r-s-1}\}$ to $\{0, \ldots, 0\}$, and $\{u_{r-s}, \ldots, u_{r-1}\}$ to $\{v_0, \ldots, v_{s-1}\}$.

Precomposition with $p_{r,s}$ determines a canonical embedding of affine schemes

$$i_{s,r} : V_s(G) \hookrightarrow V_r(G) \ ,$$

where a one-parameter subgroup $\mu : \mathbb{G}_{a(s),R} \to G_R$ of height s is sent to the one-parameter subgroup $\mu \circ p_{r,s} : \mathbb{G}_{a(r),R} \to \mathbb{G}_{a(s),R} \to G_R$ of height r. The construction is transitive, that is, we have $i_{s,r} = i_{s',r} \circ i_{s,s'}$ for $s \le s' \le r$. Hence, we have an inductive system

$$V_1(G) \subset V_2(G) \subset \ldots \subset V_r(G) \subset \ldots$$

Conversely, any one-parameter subgroup $\mathbb{G}_{a(s'),R} \to G_R$ can be decomposed as

$$\mathbb{G}_{a(s'),R} \xrightarrow{p_{s',s}} \mathbb{G}_{a(s),R} \hookrightarrow G_R$$

for some $s \leq s'$. If G is an infinitesimal group scheme of height $\leq r$ then, we must have $s \leq r$. This justifies the following definition

Definition 1.9. Let G be an infinitesimal group scheme. Then the embedding $i_{r,r'} : V_r(G) \subset V_{r'}(G)$ for $r' > r$ is an equality provided the height of G is $\leq r$. We denote by $V(G)$ the stable value of $V_r(G)$,

$$V(G) \equiv \varinjlim_r V_r(G).$$

We next recall the construction of 1-parameter subgroups for GL_n. This construction can be applied to any affine group scheme of exponential type (see [22, §1] and also [20] for an extended list of groups of exponential type). We define the homomorphism

$$\exp_{\underline{\alpha}} : \mathbb{G}_{a(r),R} \to \mathrm{GL}_{n,R}$$

of R-group schemes corresponding to an r-tuple $\underline{\alpha} = (\alpha_0, \dots, \alpha_{r-1}) \in M_n(R)^{\times r}$ of pairwise commuting p-nilpotent matrices to be the natural transformation of group-valued functors on commutative R-algebras S sending any $s \in S$ with $s^{p^r} = 0$ to

$$\exp(s\alpha_0) \cdot \exp(s^p\alpha_1) \cdot \dots \cdot \exp(s^{p^{r-1}}\alpha_{r-1}) \in \mathrm{GL}_n(S). \tag{1.9.1}$$

where for any p-nilpotent matrix $A \in \mathrm{GL}_n(S)$ we set

$$\exp(A) = 1 + A + \frac{A^2}{2} + \dots + \frac{A^{p-1}}{(p-1)!}.$$

Proposition 1.10. [22, 1.2] *The scheme of one-parameter subgroups $V_r(\mathrm{GL}_n)$ is isomorphic to the scheme of r-tuples of pairwise commuting p-nilpotent $n \times n$ matrices $N_p^{[r]}(gl_n)$; the identification is given by sending $\underline{\alpha} = (\alpha_0, \dots, \alpha_{r-1}) \in N_p^{[r]}(gl_n)(R)$ to the one-parameter subgroup* $\exp_{\underline{\alpha}} : \mathbb{G}_{a(r),R} \to \mathrm{GL}_{n,R}$.

Example 1.11. We describe $V(G)$ in each of the four examples of Example 1.4.

(1) $V(\underline{\mathfrak{g}}) \simeq N_p(\mathfrak{g})$, the closed subvariety of the affine space underlying $\mathfrak{g}$ consisting of p-nilpotent elements $x \in \mathfrak{g}$ (that is, $x^{[p]} = 0$). Let g_a be the Lie algebra of the additive group $\mathbb{G}_a$. Note that g_a is a one-dimensional restricted Lie algebra with trivial p-restriction. Each p-nilpotent element $x \in g_R = g \otimes_k R$ determines a map of p-restricted Lie algebras over R where R is a commutative k-algebra: $g_{a,R} \to g_R$. The corresponding map of height 1 infinitesimal group schemes $\mathbb{G}_{a(1),R} \to \underline{\mathfrak{g}}_R$ is the associated 1-parameter subgroup of $\underline{\mathfrak{g}}$.

(2) $V(\mathbb{G}_{a(r)}) \simeq \mathbb{A}^r$. The r-tuple $\underline{a} = (a_0, \dots, a_{r-1}) \in R^{\times r} = \mathbb{A}^r(R)$ corresponds to the 1-parameter subgroup $\mu_{\underline{a}} : \mathbb{G}_{a(r),R} \to \mathbb{G}_{a(r),R}$ whose map on coordinate algebras $R[t]/t^{p^r} \to R[t]/t^{p^r}$ sends t to $\sum_i a_i t^{p^i}$ ([22, 1.10]).

(3) By Proposition 1.10, $V(\mathrm{GL}_{n(r)}) = N_p^{[r]}(gl_n)$, the variety of r-tuples of pairwise commuting, p-nilpotent $n \times n$ matrices. The embedding $i_{r,r+1} : V_r(\mathrm{GL}_n) \simeq N_p^{[r]}(gl_n) \subset V_{r+1}(\mathrm{GL}_n) \simeq N_p^{[r+1]}(gl_n)$ described in Remark 1.8 is given by sending an r-tuple $(\alpha_0, \dots, \alpha_{r-1})$ to the $(r+1)$-tuple $(0, \alpha_0, \dots, \alpha_{r-1})$.

Let X_{ij} be the coordinate functions of $R[\mathrm{GL}_{n(r)}] \simeq R[X_{ij}]/(X_{ij}^{p^r} - \delta_{ij})$. Then $\exp^*_{\underline{\alpha}} : R[\mathrm{GL}_{n(r)}] \to R[\mathbb{G}_{a(r)}]$ is given by sending X_{ij} for some $1 \leq i, j \leq n$ to the (i,j)-entry of the polynomial $p_{\underline{\alpha}}(t)$ with matrix coefficients whose coefficient of t^d is computed as the multiple of s^d in the (i,j)-entry of the matrix (1.9.1).

Upon performing the indicated multiplication in (1.9.1), the coefficient of $p_{\underline{\alpha}}(t)$ multiplying s^{p^ℓ} is α_ℓ for $0 \leq \ell < r$, whereas coefficients of $p_{\underline{\alpha}}(t)$ multiplying s^n for n not a power of p are determined as in formula (1.6.1). Consequently, we conclude that $\exp^*_{\underline{\alpha}}(X_{ij})$ is a polynomial in t whose coefficient multiplying t^{p^ℓ} is $(\alpha_\ell)_{i,j}$ for $0 \leq \ell < r$.

(4) Since $\mathrm{SL}_{2(2)}$ is a group scheme with an embedding of exponential type (see [22, 1.8]), its variety admits a description similar to the one of $\mathrm{GL}_{n(r)}$. Namely, $V(\mathrm{SL}_{2(2)})$ is the variety of pairs of p-nilpotent proportional 2×2 matrices $\underline{\alpha} = (\alpha_0, \alpha_1)$. This variety is given explicitly as the affine scheme with coordinate algebra

$$k[V(\mathrm{SL}_{2(2)})] = k[x_0, y_0, z_0, x_1, y_1, z_1]/(x_i y_i - z_i^2, x_1 y_2 - x_2 y_1, z_1 y_2 - z_2 y_1, x_1 z_2 - x_2 z_1).$$

We give an explicit description of the map on coordinate algebras

$$\exp^*_{\underline{\alpha}} : R[\mathrm{SL}_{2(2)}] \to R[\mathbb{G}_{\mathrm{a}(2)}] \simeq R[t]/t^{p^2}$$

induced by the one-parameter subgroup $\exp_{\underline{\alpha}} : \mathbb{G}_{a(2),R} \to \mathrm{SL}_{2(2),R}$. This description follows immediately from the general discussion in the previous example. Let $\underline{\alpha} = (\begin{bmatrix} c_0 & a_0 \\ b_0 & -c_0 \end{bmatrix}, \begin{bmatrix} c_1 & a_1 \\ b_1 & -c_1 \end{bmatrix}) \in N^{[2]}(sl_2)$. Then $\exp^*_{\underline{\alpha}}$ is determined by the formulae

$$X_{11} \mapsto 1 + c_0 t + c_1 t^p, \quad X_{12} \mapsto a_0 t + a_1 t^p$$
$$X_{21} \mapsto b_0 T + b_1 t^p, \quad X_{22} \mapsto 1 - c_0 t - c_1 t^p,$$

where X_{ij} are the standard polynomial generators of $k[\mathrm{SL}_{2(2)}] \simeq \frac{k[X_{11}, X_{12}, X_{21}, X_{22}]}{(\det -1, X_{ij}^{p^2} - \delta_{ij})}$.

Remark 1.12. If $k(v)$ denotes the field of definition of the point $v \in V(G)$ for an infinitesimal group scheme G, then we have a naturally associated map $\operatorname{Spec} k(v) \to V(G)$ and, hence, an associated group scheme homomorphism over $k(v)$ (for r sufficiently large):

$$\mu_v : \mathbb{G}_{a(r),k(v)} \longrightarrow G_{k(v)} \ .$$

Note that if K/k is a field extension and $\mu : \mathbb{G}_{a(r),K} \to G_K$ is a group scheme homomorphism, then this data defines a point $v \in V(G)$ and a field embedding $k(v) \hookrightarrow K$ such that μ is obtained from μ_v via scalar extension from $k(v)$ to K.

We next recall the *rank variety* and *cohomological support variety* of a kG-module of an infinitesimal group scheme. We denote by

$$\mathrm{H}^\bullet(G,k) = \begin{cases} \mathrm{H}^*(G,k), & \text{if } p = 2, \\ \mathrm{H}^{\mathrm{ev}}(G,k) & \text{if } p > 2. \end{cases}$$

The map of R-algebras (but not of Hopf algebras for $r > 1$),

$$(1.12.1) \qquad \epsilon : R[u]/u^p \xrightarrow{u \mapsto u_{r-1}} R[u_0, \ldots, u_{r-1}]/(u_i^p) \ \simeq \ R\mathbb{G}_{a(r)} \ ,$$

makes its first appearance in the following definition and will recur throughout this paper.

Definition 1.13. Let G be a finite group scheme and M a finite dimensional kG-module. We define the cohomological support variety for M to be

$$|G|_M \ \equiv \ V(\mathrm{ann}_{\mathrm{H}^\bullet(G,k)} \mathrm{Ext}^*_{kG}(M,M)),$$

the reduced closed subscheme of $|G| = \mathrm{Spec}\, \mathrm{H}^\bullet(G,k)_{\mathrm{red}}$ given as the variety of the annihilator ideal of $\mathrm{Ext}^*_{kG}(M,M)$.

Definition 1.14. Let G be an infinitesimal group scheme and M a finite dimensional kG-module. We define the rank variety for M to be the reduced closed subscheme $V(G)_M$ whose points are given as follows:

$$V(G)_M = \{v \in V(G) : (\mu_{v,*} \circ \epsilon)^*(M_{k(v)}) \text{ is not free as a } k[u]/u^p \text{ - module } \}.$$

Proposition [23, 6.2] asserts that $V(G)_M$ is a closed subvariety of $V(G)$. A key result of [23] is the following theorem relating the scheme of 1-parameter subgroups $V(G)$ to the cohomology of G.

Theorem 1.15. *([23, 5.2, 6.8, 7.5]) Let G be an infinitesimal group scheme of height $\leq r$. There is a natural homomorphism of k-algebras*

$$\psi : \mathrm{H}^\bullet(G,k) \to k[V(G)]$$

with nilpotent kernel and image containing the p^r-th power of each element of $k[V(G)]$. Hence, the associated morphism of schemes

$$\Psi : V(G) \to \mathrm{Spec}\, \mathrm{H}^\bullet(G,k)$$

is a p-isogeny.

If M is a finite dimensional kG-module, then Ψ restricts to a homeomorphism

$$\Psi_M : V(G)_M \xrightarrow{\sim} |G|_M.$$

Furthermore, every closed conical subspace of $V(G)$ is of the form $V(G)_M$ for some finite dimensional kG-module M.

In the special case of of $G = \mathrm{GL}_{n(r)}$ the isogeny Ψ has an explicitly constructed inverse.

Theorem 1.16. *([22, 5.2]) There exists a homomorphism of k-algebras*

$$\overline{\phi} : k[V(\mathrm{GL}_{n(r)})] \to \mathrm{H}^\bullet(\mathrm{GL}_{n(r)},k)$$

such that $\psi \circ \overline{\phi}$ is the r^{th} iterate of the k-linear Frobenius map. Hence, the associated morphisms of schemes

$$\Psi : V(\mathrm{GL}_{n(r)}) \to \mathrm{Spec}\, \mathrm{H}^\bullet(\mathrm{GL}_{n(r)},k), \quad \Phi : \mathrm{Spec}\, \mathrm{H}^\bullet(\mathrm{GL}_{n(r)},k) \to V(\mathrm{GL}_{n(r)})$$

are mutually inverse homeomorphisms.

Example 1.17. We investigate $V(G)_M$ for the four examples of Example 1.4.

(1) Let M be a p-restricted $\mathfrak{g}$-module of dimension m, given by the map of p-restricted Lie algebras $\rho : \mathfrak{g} \to \mathrm{End}_k(M) \simeq \mathfrak{gl}_m$. Then $V(\underline{\mathfrak{g}})_M \subset V(\underline{\mathfrak{gl}_m})$ consists of those p-nilpotent elements of $\mathfrak{g}$ whose Jordan type (as an $m \times m$-matrix in $\mathfrak{gl}_m$) has at least one block of size $< p$ (see [10]).

(2) For $G = \mathbb{G}_{a(r)}$, $kG \simeq kE$ where E is an elementary abelian p-group of rank r. The rank variety of a kE-module was first investigated in [4].

We consider directly the rank variety $V(\mathbb{G}_{a(r)})_M$ of a finite dimensional $k\mathbb{G}_{a(r)}$-module M. The data of such a module is the choice of r p-nilpotent endomorphisms $\widetilde{u}_0, \ldots, \widetilde{u}_{r-1} \in \mathrm{End}_k(M)$, given as the image of the distinguished generators of $k\mathbb{G}_{a(r)}$ as in (1.4.1). A 1-parameter subgroup of $\mathbb{G}_{a(r)}$ has the form $\mu_{\underline{a}} : \mathbb{G}_{a(r),K} \to \mathbb{G}_{a(r),K}$ for some r-tuple $\underline{a} = (a_0, \ldots, a_{r-1})$ of K-rational points as in Example 1.11(2). The condition that $\mu_{\underline{a}}$ be a point of $V(\mathbb{G}_{a(r)})_M$ is the condition that $(\mu_{\underline{a}} \circ \epsilon)^*(M_K)$ is not free as a $K[u]/u^p$-module, where $u = a_{r-1}\widetilde{u}_0 + a_{r-2}^p \widetilde{u}_1 \cdots + a_0^{p^{r-1}} \widetilde{u}_{r-1} \in \mathrm{End}_K(M_K)$ (see [22, 6.5]).

(3) Let M be a finite dimensional kG-module with $G = \mathrm{GL}_{n(r)}$. By Theorem 1.15, $V(\mathrm{GL}_{n(r)})_M \subset V(\mathrm{GL}_{n(r)})$ is the closed subvariety whose set of points in a field K/k are 1-parameter subgroups : $\exp_{\underline{\alpha}} : \mathbb{G}_{a(r),K} \to \mathrm{GL}_{n(r),K}$ indexed by r-tuples $\underline{\alpha} = (\alpha_0, \ldots, \alpha_{r-1}) \in M_n(K)$ of p-nilpotent, pairwise commuting matrices such that $(\exp_{\underline{\alpha},*} \circ \epsilon)^*(M_K)$ is not a free as a $K[u]/u^p$-module. The action of u on M_K is determined utilizing Example 1.11(3). Namely, the action of u is given by composing the coproduct $M_K \to K[\mathrm{GL}_{n(r)}] \otimes M_K$ defining the $\mathrm{GL}_{n(r)}$-module structure on M_K with the linear functional $\epsilon \circ \exp_{\underline{\alpha}}^* : K[\mathrm{GL}_{n(r)}] \to K$. In §3, we shall investigate this case in more detail by considering some concrete examples.

(4) A complete description of support varieties for simple modules for $\mathrm{SL}_{2(r)}$ can be found in [23, §7]. We describe the situation for $G = \mathrm{SL}_{2(2)}$. Let S_λ be irreducible modules of highest weight λ, where $0 \leq \lambda \leq p^2 - 1$. For $\lambda < p - 1$, the module S_λ has dimension less than p and thus $V(G)_{S_\lambda} = V(G)_{S_\lambda^{(1)}} = V(G)$. For $\lambda = p - 1$, the restriction of S_{p-1} to $\mathrm{SL}_{2(1)} \subset \mathrm{SL}_{2(2)}$ is projective (the Steinberg module for $SL_{2(1)}$) but S_{p-1} is not itself projective. Hence, $V(G)_{S_{p-1}}$ is a proper non-trivial subvariety of $V(G)$. Using the notation introduced in Example 1.11(4), we have

$$V(G)_{S_{p-1}} = \{(\alpha_0, 0) \mid \alpha_0 \in N(sl_2)\} \subset V(G),$$

and

$$V(G)_{S_{p-1}^{(1)}} = \{(0, \alpha_1) \mid \alpha_1 \in N(sl_2)\} \subset V(G)$$

(see [23, 6.10]). $V(G)_{S_{p-1}}$ can be described as a subscheme of $V(G)$ defined by the equations $x_2 = y_2 = 0$. For $\lambda = \lambda_0 + \lambda_1 p$ where $\lambda_0, \lambda_1 \leq p - 1$ we have $S_\lambda \simeq S_{\lambda_0} \otimes S_{\lambda_1}^{(1)}$ by the Steinderg tensor product theorem. Hence, we can compute the support variety of S_λ using the tensor product property of support varieties. For $\lambda = p^2 - 1$, S_λ is the Steinberg module for $\mathrm{SL}_{2(2)}$, it is projective and, hence, $V(G)_{S_{p^2-1}} = \{0\}$. Overall, we get

$$V(G)_{S_\lambda} = \begin{cases} N^{[2]}(sl_2), & \text{if } \lambda_0, \lambda_1 \neq p-1, \\ \{(\alpha_0, 0) \mid \alpha_0 \in N(sl_2)\} & \text{if } \lambda_0 = p-1, \lambda_1 \neq p-1, \\ \{(0, \alpha_1) \mid \alpha_1 \in N(sl_2)\} & \text{if } \lambda_0 \neq p-1, \lambda_1 = p-1, \\ 0 & \text{if } \lambda = p^2 - 1. \end{cases}$$

2. Universal p-nilpotent operators

Let G be an affine group scheme over k. A k-linear functional with values in a commutative k-algebra A, $\Theta : k[G] \to A$, determines for every G_A-module N an

A-linear map

$$N \xrightarrow{\nabla_N} A[G] \otimes_A N \simeq k[G] \otimes N \xrightarrow{\Theta \otimes 1_N} A \otimes N \longrightarrow N$$

where $\nabla_N : N \to A[G] \otimes_A N$ is the comodule map on N. Hence, we can view such a functional $\Theta : k[G] \to A$ as an A-linear operator on a G_A-module N. Moreover, if $A \to A'$ is a homomorphism of commutative k-algebras, then Θ determines by base change an operator on $G_{A'}$-modules. If M is a G-module, then applying the above construction to $A \otimes M$, we get an action of Θ on $A \otimes M$ which is an A-linear extension of the map

$$M \xrightarrow{\nabla_M} k[G] \otimes M \xrightarrow{\Theta \otimes 1_M} A \otimes M \ .$$

Let G is a finite group scheme, so that $k[G]$ is finite dimensional with k-linear dual kG. Then we have an isomorphism $\mathrm{Hom}_k(k[G], A) \simeq kG \otimes A$. Hence, we can view a linear functional $\Theta \in \mathrm{Hom}_k(k[G], A)$ as an element of $A \otimes kG \equiv AG$, so that the action of Θ for a given kG-module M is given by multiplication by $\Theta \in AG$, $\Theta : M \otimes A \to \ M \otimes A$.

Definition 2.1. Let G be an affine group scheme over k and $\Theta : k[G] \to A$ a k-linear functional for some commutative k-algebra A. Then Θ is said to be p-nilpotent if the composition

$$k[G] \xrightarrow{\quad (\nabla_G)^{\otimes p} \quad} k[G]^{\otimes p} \xrightarrow{\quad \Theta^{\otimes p} \quad} A$$

is 0. We view such a p-nilpotent functional Θ as a p-nilpotent operator on G_A-modules.

If A is graded and if $\Theta(k[G])$ lies in the d^{th}-graded summand of A, then Θ is said to be homogeneous of degree d.

The universal p-nilpotent operator we consider arises as follows. Let G be an affine group scheme over k. The natural isomorphism of covariant functors on commutative k-algebras R

$$\mathrm{Hom}_{\mathrm{grp\ sch}}(\mathbb{G}_{a(r),R}, G_R) \ \simeq \mathrm{Hom}_{k-\mathrm{alg}}(k[V_r(G)], R)$$

given in Theorem 1.7 implies the existence of a universal 1-parameter subgroup of height r

$$\mathcal{U}_{G,r} : \mathbb{G}_{a(r),k[V_r(G)]} \ \longrightarrow \ G_{k[V_r(G)]}.$$

The subgroup $\mathcal{U}_{G,r}$ induces a map on coordinate algebras

$$\mathcal{U}^*_{G,r} : k[V_r(G)] \otimes k[G] \longrightarrow k[V_r(G)] \otimes k[\mathbb{G}_{a(r)}].$$

If G is a finite group scheme, then $\mathcal{U}_{G,r}$ induces a map on group algebras over $k[V_r(G)]$:

$$\mathcal{U}_{G,r,*} : k[V_r(G)] \otimes k\mathbb{G}_{a(r)} \longrightarrow k[V_r(G)] \otimes kG.$$

Recall that $k\mathbb{G}_{a(r)} \simeq k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p)$, so that u_{r-1}, the dual element to $T^{p^{r-1}}$, can be viewed as a linear map $u_{r-1} : k[\mathbb{G}_{a(r)}] \to k$.

Definition 2.2. Let G be an affine group scheme over k. We define the universal p-nilpotent operator for G,

$$\Theta_{G,r} : k[G] \ \longrightarrow \ k[V_r(G)]$$

to be the k-linear, p-nilpotent functional defined by the composition

$$k[G] \xrightarrow{1\otimes \mathrm{id}} k[V_r(G)] \otimes k[G] \xrightarrow{\mathcal{U}^*_{G,r}} k[V_r(G)] \otimes k[\mathbb{G}_{a(r)}] \xrightarrow{u_{r-1}} k[V_r(G)]. \tag{2.2.1}$$

Remark 2.3. For a finite group scheme G, the operator $\Theta_{G,r} \in \mathrm{Hom}_k(k[G], k[V_r(G)]) \simeq k[V_r(G)] \otimes kG$ can be equivalently defined as the image of u under the composition

$$k[u]/u^p \xrightarrow{\ \epsilon\ } k[V_r(G)] \otimes k\mathbb{G}_{a(r)} \xrightarrow{\mathcal{U}_{G,r,*}} k[V_r(G)] \otimes kG, \tag{2.3.1}$$

where $\epsilon : k[u]/u^p = k\mathbb{G}_{a(1)} \to k\mathbb{G}_{a(r)} = k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p)$ is the map of k-algebras that sends u to u_{r-1} (see (1.12.1)).

For a G-module M, $\Theta_{G,r} \in k[V_r(G)] \otimes kG$ determines a $k[V_r(G)]$-linear p-nilpotent operator

$$\Theta_{G,r} : k[V_r(G)] \otimes M \longrightarrow k[V_r(G)] \otimes M \tag{2.3.2}$$

$$a \otimes m \mapsto \Theta_{G,r}(a \otimes m) = a\Theta_{G,r}(1 \otimes m).$$

If G is infinitesimal of height $\leq r$, then by Remark 1.8, $V_r(G)$ is essentially independent of r. The following proposition justifies our using the simplified notation

$$\mathcal{U}_G : \mathbb{G}_{a(r),k[V(G)]} \ \to \ G_{k[V(G)]}, \quad \Theta_G \ \in \ k[V(G)] \otimes kG.$$

Proposition 2.4. *Let G be an affine group scheme and let $r' \geq r$. For notational convenience, set $A_r = k[V_r(G)]$, $A_{r'} = k[V_{r'}(G)]$. Let $i^* : A_{r'} \twoheadrightarrow A_r$ be the projection corresponding to the embedding $i_{r,r'} : V_r(G) \hookrightarrow V_{r'}(G)$ which is induced by the canonical projection $p_{r',r} : \mathbb{G}_{a(r')} \twoheadrightarrow \mathbb{G}_{a(r)}$ (see Remark 1.8). Consider A_r as an $A_{r'}$-module via i^*.*

If G is an infinitesimal group scheme, then

$$\Theta_{G,r} = \Theta_{G,r'} \otimes_{A_{r'}} 1 \ \in \ A_{r'}G \otimes_{A_{r'}} A_r \simeq A_rG.$$

Moreover, if G is an infinitesimal group scheme of height $\leq r$, then $\Theta_{G,r}$ is thereby naturally identified with $\Theta_{G,r'}$.

Proof. Consider the composition

$$\begin{tikzcd} \mathbb{G}_{a(r'),A_r} \arrow[rr, "\mathcal{U}_{G,r} \circ p_{r',r}"] \arrow[dr, "p_{r',r}"'] & & G_{A_r} \\ & \mathbb{G}_{a(r),A_r} \arrow[ur, "\mathcal{U}_{A_r}"'] & \end{tikzcd} \tag{2.4.1}$$

Since $\mathcal{U}_{G,r} \in V_r(G)(A_r) \simeq \mathrm{Hom}(A_r, A_r)$ corresponds to the identity map on A_r, and $p_{r',r}$ is the map that induces $i^* : A_{r'} \to A_r$, we conclude that the composition $\mathcal{U}_{G,r} \circ p_{r',r} \in V_{r'}(G) \simeq \mathrm{Hom}(A_{r'}, A_r)$ corresponds to i^*. Hence, the universality of $\mathcal{U}_{G,r'}$ implies that $\mathcal{U}_{G,r} \circ p_{r',r}$ is obtained by pulling back the universal one-parameter subgroup $\mathcal{U}_{G,r'}$ via $i^* : A_{r'} \to A_r$. Therefore, we conclude

$$\mathcal{U}_{G,r} \circ p_{r',r} \ = \ \mathcal{U}_{G,r'} \otimes_{A_{r'}} A_r \tag{2.4.2}$$

which implies the equality of maps of group algebras

$$\mathcal{U}_{G,r'*} \otimes_{A_{r'}} A_r = \mathcal{U}_{G,r*} \circ p_{r',r,*} : \ A_r\mathbb{G}_{a(r')} \to A_rG. \tag{2.4.3}$$

Since $p_{r',r,*}(u_{r'}) = u_r \in k\mathbb{G}_{a(r)}$, we conclude $(\mathcal{U}_{G,r*} \circ p_{r',r,*})(u_{r'}) = \mathcal{U}_{G,r*}(u_r) = \Theta_{G,r}$, whereas $(\mathcal{U}_{G,r'*} \otimes_{A_{r'}} A_r)(u_{r'}) = \mathcal{U}_{G,r'*}(u_{r'}) \otimes_{A_{r'}} 1 = \Theta_{G,r'} \otimes_{A_{r'}} 1$.

The second statement follows immediately from the fact that for G of height $\leq r$, the map $i^* : A_{r'} \to A_r$ is an isomorphism as shown in Remark 1.8. $\square$

Example 2.5. We describe the universal p-nilpotent operator Θ_G in each of the four examples of Example 1.4.

(1) Let $G = \underline{\mathfrak{g}}$ for some finite dimensional p-restricted Lie algebra $\mathfrak{g}$ embedded as a p-restricted subalgebra of some $\mathfrak{gl}_m$. Then $k[V(\underline{\mathfrak{g}})] = k[N_p(\mathfrak{g})]$, which we view as a quotient of $k[X_{ij}]$ (where $1 \leq i, j \leq m$) by the ideal of relations that a general $m \times m$ matrix must satisfy to be a p-nilpotent element of $\mathfrak{g} \subset \mathfrak{gl}_m$.

Then $\Theta_{\underline{\mathfrak{g}}} \in k[N_p(\mathfrak{g})] \otimes \mathfrak{u}(\mathfrak{g})$ has image in $k[N_p(\mathfrak{g})] \otimes \mathfrak{u}(\mathfrak{gl}_m)$ equal to the image of $\Theta_{\underline{\mathfrak{gl}_m}} \in k[N_p(\mathfrak{gl}_m)] \otimes u(\mathfrak{gl}_m)$. Moreover, $\Theta_{\underline{\mathfrak{gl}_m}}$ is given explicitly as the image of the generic matrix in $k[X_{ij}] \otimes \mathfrak{u}(\mathfrak{gl}_m)$.

In other words,

$$\Theta_{\underline{\mathfrak{g}}} \; = \; \sum_{x_i} \check{x}_i \otimes x_i \;\; \in \;\; k[N_p(\mathfrak{g})] \otimes \mathfrak{u}(\mathfrak{g}) \tag{2.5.1}$$

where $\{x_i\}$ is a basis of $\mathfrak{g}$ and $\check{x}_i$ denotes the image of the dual basis element to x_i under the quotient map $S^*(\mathfrak{g}^{\#}) \to k[N_p(\underline{\mathfrak{g}})]$.

We record an explicit formula for the universal p-nilpotent operator in the case of $\mathfrak{g} = sl_2$ for future reference. We have $k[N_p(sl_2)] \simeq k[x, y, z]/(xy + z^2)$. Let e, f, h be the standard basis of the p-restricted Lie algebra sl_2. Then

$$\Theta_{\underline{sl}_2} = xe + yf + zh.$$

Observe that this formula agrees with the presentation of a "generic" π-point for $u(sl_2)$ as given in [14, 2.5].

(2) Take $G = \mathbb{G}_{a(r)}$. Then $k[\mathbb{G}_{a(r)}] \simeq k[T]/T^{p^r}$, and $k[V(\mathbb{G}_{a(r)})] \simeq k[x_0, \ldots, x_{r-1}]$ is graded in such a way that x_i has degree p^i (see Proposition 2.8 below). We compute $\Theta = \Theta_{\mathbb{G}_{a(r)}} : k[\mathbb{G}_{a(r)}] \longrightarrow k[V(\mathbb{G}_{a(r)})]$ explicitly in this case (see also [23, 6.5.1]).

One-parameter subgroups of $\mathbb{G}_{a(r),K}$ are in one-to-one correspondence with the additive polynomials in $K[T]/T^{p^r}$, that is, polynomials of the form $p(T) = a_0T + a_1T^p + \ldots + a_{r-1}T^{p^{r-1}}$ (see [22, 1.10]). The map on coordinate algebras induced by the universal one-parameter subgroup $\mathcal{U} : \mathbb{G}_{a(r),k[V(G)]} \to \mathbb{G}_{a(r),k[V(G)]}$ is given by the "generic" additive polynomial:

$$\mathcal{U}^* : k[x_0, \ldots, x_{r-1}][T]/T^{p^r} \longrightarrow k[x_0, \ldots, x_{r-1}][T]/T^{p^r} .$$

$$T \mapsto x_0T + x_1T^p + \ldots + x_{r-1}T^{p^{r-1}}.$$

To determine the linear functional

$$\Theta = u_{r-1} \circ \mathcal{U}^* : k[T]/T^{p^r} \longrightarrow k[x_0. \ldots, x_{r-1}] \; ,$$

it suffices to determine the values of Θ on the linear generators $\{T^i\}, 0 \leq i \leq p^r - 1$. Since u_{r-1} is the dual to $T^{p^{r-1}}$, this further reduces to determining the coefficient by $T^{p^{r-1}}$ in $U^*(T^i) = (x_0T + x_1T^p + \ldots + x_{r-1}T^{p^{r-1}})^i$. Computing this coefficient, we conclude that Θ is given explicitly on the basis elements of $k[\mathbb{G}_{a(r)}] \simeq k[T]/T^{p^r}$

by

$$T^i \mapsto \sum_{\substack{i_0+i_1+\cdots+i_{r-1}=i \\ i_0+i_1p+\cdots+i_{r-1}p^{r-1}=p^{r-1}}} \binom{i}{i_0, i_1, \ldots, i_{r-1}} x_0^{i_0} \ldots x_{r-1}^{i_{r-1}}, \tag{2.5.2}$$

where $\binom{i}{i_0,i_1,\ldots,i_{r-1}} = \frac{i!}{i_0!i_1!\ldots i_{r-1}!}$ is the multinomial coefficient. Let $\{v_0, \ldots, v_{p^r-1}\}$ be the linear basis of $k\mathbb{G}_{a(r)}$ dual to $\{1, T, \ldots, T^{p^r-1}\}$. Dualizing (2.5.2), we obtain that, as an element of $k[V(\mathbb{G}_{a(r)})] \otimes k\mathbb{G}_{a(r)}$, Θ has the following form:

$$\Theta = \sum_{i=0}^{p^r-1} \left[\sum_{\substack{i_0+i_1+\cdots+i_{r-1}=i \\ i_0+i_1p+\cdots+i_{r-1}p^{r-1}=p^{r-1}}} \binom{i}{i_0, i_1, \ldots, i_{r-1}} x_0^{i_0} \ldots x_{r-1}^{i_{r-1}} \right] v_i. \tag{2.5.3}$$

Finally, we recall that v_i can expressed in terms of the algebraic generators u_j of $k\mathbb{G}_{a(r)}$ via the following formulae ([22, 1.14])

$$v_i = \frac{u_0^{i^{(0)}} \cdot \ldots \cdot u_{r-1}^{i^{(r-1)}}}{i^{(0)}! \cdot \ldots \cdot i^{(r-1)}!},$$

where $i = i^{(0)} + i^{(1)}p + \ldots + i^{(r-1)}p^{r-1} \quad (0 \leq i^{(j)} \leq p-1)$ is the p-adic expansion of i.

(3) Let $G = \mathrm{GL}_{n(r)}$. Recall that $V(G)$ is the scheme of r-tuples of p-nilpotent, pair-wise commuting matrices. For notational convenience, let A denote $k[V(G)]$. Then $\mathcal{U}_{\mathrm{GL}_{n(r)}} : \mathbb{G}_{a(r),A} \to \mathrm{GL}_{n(r),A}$ is specified by the A-linear map on coordinate algebras

$$\mathcal{U}^*_{\mathrm{GL}_{n(r)}} : A[\mathrm{GL}_{n(r)}] \ \to \ A[T]/T^{p^r}, \quad X_{a,b} \mapsto \sum_{j=0}^{p^r-1} (\beta_j)_{a,b} T^j$$

where $\{X_{a,b}; 1 \leq a, b \leq n\}$ are the coordinate functions of GL_n, where β_j is given as in formula (1.6.1) in terms of the matrices $\alpha_0, \ldots, \alpha_{r-1} \in M_n(A)$, and $\alpha_i = \beta_{p^i}$ have matrix coordinate functions which generate A. (In other words, the $n^2 r$ entries of $\alpha_0, \ldots, \alpha_{r-1}$ viewed as variables generate A, with relations given by the conditions that these matrices must be p-nilpotent and pairwise commuting.)

The p-nilpotent operator

$$\Theta_{\mathrm{GL}_{n(r)}} \ = \ (\mathcal{U}_{\mathrm{GL}_{n(r)}\,*} \circ \epsilon)(u) \ \in \ \mathrm{Hom}_k(k[\mathrm{GL}_{n(r)}], A) = k[\mathrm{GL}_{n(r)}] \otimes k\,\mathrm{GL}_{n(r)}$$

is given by the k-linear functional sending a polynomial in the matrix coefficients $P(X_{a,b}) \in k[\mathrm{GL}_{n(r)}]$ to the coefficient of $T^{p^{r-1}}$ of the sum of products corresponding to the polynomial P given by replacing each $X_{a,b}$ by $\sum_{j=0}^{p^r-1} (\beta_j)_{a,b} T^j$ (when taking products of matrix coefficients, one uses the usual rule for matrix multiplication).

The coaction $k^n \to k^n \otimes k[\mathrm{GL}_n]$ corresponding to the natural representation of GL_n on k^n determines an action of $\mathrm{Hom}_k(k[\mathrm{GL}_{n(r)}], A) \subset \mathrm{Hom}_k(k[\mathrm{GL}_n], A)$ on A^n, so that we may associate to Θ_G an A-linear endomorphism of A^n given in matrix form by $(\Theta_G(X_{a,b}))$.

(4) We compute $\Theta_{\mathrm{SL}_{2(2)}}$ explicitly. Recall that $V(\mathrm{SL}_{2(2)})$ is the variety of pairs of commuting, trace 0, nilpotent matrices with coordinate algebra $A = k[V(\mathrm{SL}_{2(2)})]$ as determined in Example 1.11(4). A 1-parameter subgroup $\mathbb{G}_{a(2),R} \to \mathrm{SL}_{2(2),R}$

is specified by a map on coordinate algebras $R[\mathrm{SL}_{2(2)}] \to R[t]/t^{p^2}$ as described in Example 1.11(4).

As in Example 1.4(4), write $e, f, h, e^{(p)}, f^{(p)}, h^{(p)}$ for the generators of $k\,\mathrm{SL}_{2(2)}$ and set

$$e^{(i)} = \frac{e^i}{i!}, \quad f^{(i)} = \frac{f^i}{i!}, \quad \binom{h}{i} = \frac{h(h-1)(h-2)\dots(h-i+1)}{i!}$$

for $i < p$. Fix the linear basis of $k[\mathrm{SL}_2(2)]$ given by powers of $X_{12}, X_{21}, X_{11} - 1$ (in this fixed order). Then the element of $k\,\mathrm{SL}_{2(2)}$ dual to $X_{12}^i X_{21}^j (X_{11} - 1)^\ell$ for $i + j + \ell \leq p$ is given by

$$(X_{12}^i X_{21}^j (X_{11} - 1)^\ell)^\# \;=\; e^{(i)} f^{(j)} \binom{h}{\ell}$$

(where $\binom{h}{p}$ is identified with $h^{(p)}$ by definition).
With these conventions $\Theta_{\mathrm{SL}_{2(2)}} \in k[V(\mathrm{SL}_{2(2)})] \otimes k\,\mathrm{SL}_{2(2)}$ equals

$$x_1 e + y_1 f + z_1 h + x_0^p e^{(p)} + y_0^p f^{(p)} + z_0^p h^{(p)} + \sum_{\substack{i+j+\ell=p \\ i,j,\ell<p}} x_0^i y_0^j z_0^\ell e^{(i)} f^{(j)} \binom{h}{\ell}. \tag{2.5.4}$$

To complement Example 2.5, we make explicit the action of Θ_G on some kG-representation for each of the four types of finite group schemes we have been considering in examples.

Example 2.6. (1) Let $G = \underline{\mathfrak{g}}$ and let $M = \mathfrak{g}^{ad}$ denote the adjoint representation of the p-restricted Lie algebra $\mathfrak{g}$; let $\{x_i\}$ be a basis for $\mathfrak{g}$. We identify $\Theta_{\underline{\mathfrak{g}}}$ as the $k[N_p(\mathfrak{g})]$-linear endomorphism

$$\Theta_{\underline{\mathfrak{g}}} : k[N_p(\mathfrak{g})] \otimes \mathfrak{g}^{ad} \;\to\; k[N_p(\mathfrak{g})] \otimes \mathfrak{g}^{ad}, \quad 1 \otimes x \mapsto \sum_i \check{x}_i \otimes [x_i, x],$$

where $\check{x}_i$ is the image under the projection $S^*(\mathfrak{g}^\#) \to k[N_p(\mathfrak{g})]$ of the dual basis element to x_i.

(2) Let M denote the cyclic $k\mathbb{G}_{a(r)}$-module

$$M \;=\; k[u_0, \dots, u_{r-1}]/(u_0, u_1^p, \dots, u_{p-1}^p) = k[u_1, \dots, u_{r-1}]/(u_1^p, \dots, u_{p-1}^p).$$

As recalled in Example 2.5(2), $k[V(\mathbb{G}_{a(r)})] = k[\mathbb{A}^r] = k[a_0, \dots, a_{r-1}]$, $k\mathbb{G}_{a(r)} = k[u_0, \dots, u_{r-1}]/(u_i^p)$, and

$$\Theta_{\mathbb{G}_{a(r)}} \in \; A \otimes k[u_0, \dots, u_{r-1}]/(u_i^p)$$

is given by the complicated, but explicit formula (2.5.3). We conclude that

$$\Theta_{\mathbb{G}_{a(r)}} : A \otimes M \;\to\; A \otimes M$$

is the A-linear endomorphism sending u_i to $\overline{\Theta}_{\mathbb{G}_{a(r)}} \cdot u_i$, where $\overline{\Theta}_{\mathbb{G}_{a(r)}}$ is the image of $\Theta_{\mathbb{G}_{a(r)}}$ under the projection $A \otimes k[u_0, \dots, u_{r-1}]/(u_i^p) \to A \otimes M$.

(3) Let M be the restriction to $\mathrm{GL}_{n(r)}$ of the canonical n-dimensional rational GL_n-module V_n. By Example 1.11(3), $A = k[V(\mathrm{GL}_{n(r)}]$ is the quotient of $k[gl_n]^{\otimes r}$ by the ideal generated by the equations satisfied by an r-tuple of $n \times n$-matrices

with the property that each matrix is p-nilpotent and that the matrices pair-wise commute. The complexity of the map

$$\Theta_{\mathrm{GL}_{n(r)}} : A \otimes V_n \ \to\ A \otimes V_n$$

is revealed even in the case $n = 2$ which is worked out explicitly below.

(4) Let M be the restriction to $\mathrm{SL}_{2(2)}$ of the rational GL_2 representation V_2. Then Example 1.11(4) gives an explicit description of $A = k[V(\mathrm{SL}_{2(2)})]$ as a quotient of $k[x_0, y_0, z_0, x_1, y_1, z_1]$ and (2.5.4) gives $\Theta_{\mathrm{SL}_{2(2)}}$ explicitly. The divided powers $e^{(p)}$, $f^{(p)}$ and $h^{(p)}$ as well as all products of the form $e^{(i)} f^{(j)} \binom{h}{\ell}$ act trivially on M. Hence, the map

$$\Theta_{\mathrm{SL}_{2(2)}} : A \otimes M \ \to\ A \otimes M$$

is given by the matrix

$$A^2 \xrightarrow{\begin{bmatrix} z_1^p & x_1^p \\ y_1^p & -z_1^p \end{bmatrix}} A^2.$$

The following naturality property of Θ_G will prove useful when we consider $M \otimes k[V(G)]$ as a free, coherent sheaf on $V(G)$ and restrict this sheaf to $V(H) \subset V(G)$ equipped with its action of H.

When viewing group schemes as functors, it is often convenient to think of $G_{k[V(G)]}$ as $G \times V(G)$ (i.e., $G \times V(G) = \mathrm{Spec}(k[V(G)] \otimes k[G])$). From this point of view, $\mathcal{U}_G$ has the form

$$\mathcal{U}_G : \mathbb{G}_{a(r)} \times V(G) \ \longrightarrow\ G \times V(G).$$

Proposition 2.7. *Let $i : H \hookrightarrow G$ be a closed embedding of affine group schemes over k inducing the map $\phi : V_r(H) \to V_r(G)$ for some given $r > 0$. Let $\phi^* : k[V_r(G)] \to k[V_r(H)]$ denote the map on coordinate algebras associated to ϕ. Then the following square commutes*

(2.7.1)
$$\begin{tikzcd} \mathbb{G}_{a(r)} \times V_r(H) \arrow[r, "\mathcal{U}_{H,r}"] \arrow[d, "id\times\phi"'] & H \times V_r(H) \arrow[d, "i\times\phi"] \\ \mathbb{G}_{a(r)} \times V_r(G) \arrow[r, "\mathcal{U}_{G,r}"] & G \times V_r(G). \end{tikzcd}$$

Consequently, the following square of k-linear maps commutes:

(2.7.2)
$$\begin{tikzcd} k[G] \arrow[r, "\Theta_{G,r}"] \arrow[d, "i^*"] & k[V_r(G)] \arrow[d, "i^*"] \\ k[H] \arrow[r, "\Theta_{H,r}"] & k[V_r(H)]. \end{tikzcd}$$

Thus, for any rational G-module M we have a compatibility of coactions on M:

(2.7.3)
$$\begin{tikzcd} M \arrow[r, "\nabla_M"] \arrow[dr, "\nabla_{M\downarrow H}"'] & k[G] \otimes M \arrow[r, "\Theta_{G,r}\otimes 1"] \arrow[d, "i^*"] & k[V_r(G)] \otimes M \arrow[d, "i^*"] \\ & k[H] \otimes M \arrow[r, "\Theta_{H,r}\otimes 1"] & k[V_r(H)] \otimes M. \end{tikzcd}$$

Proof. By universality of $\mathcal{U}_{G,r}$, the composition $(i \times id) \circ \mathcal{U}_{H,r} : \mathbb{G}_{a(r)} \times V_r(H) \to G \times V_r(H)$ is obtained by pull-back of $\mathcal{U}_{G,r}$ via some morphism $V_r(H) \to V_r(G)$. By comparing maps on R-valued points, we verify that this morphism must be ϕ. This implies the commutativity of (2.7.1).

The commutative square (2.7.1) gives a commutative square on coordinate algebras:

$$\begin{CD} k[V_r(G)] \otimes k[G] @>{\mathcal{U}^*_{G,r}}>> k[V_r(G)] \otimes k[\mathbb{G}_{a(r)}] \\ @V{\phi^* \otimes i^*}VV @VV{\phi^* \otimes \mathrm{id}}V \\ k[V_r(H)] \otimes k[H] @>{\mathcal{U}^*_{H,r}}>> k[V_r(H)] \otimes k[\mathbb{G}_{a(r)}]. \end{CD} \tag{2.7.4}$$

Concatenating (2.7.4) on the right with the commutative square of linear maps

$$\begin{CD} k[V_r(G)] \otimes k[\mathbb{G}_{a(r)}] @>{u_{r-1}}>> k[V_r(G)] \\ @V{\phi^* \otimes \mathrm{id}}VV @VV{\phi^* \otimes \mathrm{id}}V \\ k[V_r(H)] \otimes k[\mathbb{G}_{a(r)}] @>{u_{r-1}}>> k[V_r(H)] \end{CD}$$

and with the inclusions $k[G] \to k[V_r(G)] \otimes k[G]$ and $k[H] \to k[V_r(H)] \otimes k[H]$ on the left, we obtain a commutative diagram:

$$\begin{CD} k[G] @>>> k[V_r(G)] \otimes k[G] @>{\mathcal{U}^*_{G,r}}>> k[V_r(G)] \otimes k[\mathbb{G}_{a(r)}] @>{u_{r-1}}>> k[V_r(G)] \\ @V{i^*}VV @V{\phi^* \otimes i^*}VV @V{\phi^* \otimes \mathrm{id}}VV @V{\phi^* \otimes \mathrm{id}}VV \\ k[H] @>>> k[V_r(H)] \otimes k[H] @>{\mathcal{U}^*_{H,r}}>> k[V_r(H)] \otimes k[\mathbb{G}_{a(r)}] @>{u_{r-1}}>> k[V_r(H)]. \end{CD}$$

Eliminating the middle square, we obtain the square (2.7.2). Hence, it is commutative.

Finally, the commutativity of (2.7.3) follows immediately from the commutativity of (2.7.2). $\square$

For any affine group scheme G, the k-algebra $k[V_r(G)]$ is provided with a natural grading determined by the action of $\mathbb{A}^1 \simeq V_r(\mathbb{G}_{a(1)}) \subset V_r(\mathbb{G}_{a(r)})$ on $V_r(G)$ (see [22, 1.12]). From the point of view of functors on commutative k-algebra R, this grading is determined by pre-composition

$$V_r(G)(R) \times V_r(\mathbb{G}_{a(1)})(R) \ \to \ V_r(G)(R) \times V_r(\mathbb{G}_{a(r)})(R) \ \to \ V_r(G)(R).$$

Observe that this grading is functorial with respect to homomorphisms $G \to G'$ of group schemes.

If $G = \mathrm{GL}_N$, then an R-valued point of $V_r(\mathrm{GL}_N)$ is given by an r-tuple of $N \times N$ pair-wise commuting, p-nilpotent matrices with entries in R, $(\alpha_0, \dots, \alpha_{r-1})$. The coordinate functions of the matrix α_i have grading p^i; in other words, the action of $c \in V(\mathbb{G}_{a(1)})(R)$ on $(\alpha_0, \alpha_1, \dots, \alpha_{r-1}) \in V(\mathrm{GL}_{N(r)})(R)$ is given by the formula

$$c \cdot (\alpha_0, \alpha_1, \dots, \alpha_{r-1}) = (c\alpha_0, c^p \alpha_1, \dots, c^{p^{r-1}} \alpha_{r-1}).$$

More generally, if G is a closed subgroup scheme of GL_N of height $\leq r$, then the embedding $G \subset \mathrm{GL}_N$ induces $V(G) \to V(\mathrm{GL}_{N(r)})$ whose associated map on coordinate algebras $k[V(G)] \leftarrow k[V(\mathrm{GL}_{N(r)})]$ is a map of graded algebras.

Proposition 2.8. *For any affine group scheme G and integer $r > 0$, the k-linear map*

$$\Theta_{G,r} : k[G] \ \to \ k[V_r(G)]$$

has image contained in the homogeneous summand of $k[V_r(G)]$ of degree p^{r-1}.

In other words,

$$\mathcal{U}_{G,r,*} : k[V_r(G)] \otimes k\mathbb{G}_{a(r)} \ \to \ k[V_r(G)] \otimes kG$$

sends $1 \otimes u_{r-1} \in k[V_r(G)] \otimes k\mathbb{G}_{a(r)}$ to some $\sum a_i \otimes x_i \in k[V_r(G)] \otimes kG$ with each $a_i \in k[V_r(G)]$ homogeneous of degree p^{r-1}.

Proof. Let $A = k[V_r(G)]$. Let $\langle \lambda_i \rangle$ be a set of linear generators of $k[G]$, and $\langle \check{\lambda}_i \rangle$ be the dual set of linear generators of kG. Then $\mathcal{U}_{G,r,*}(u_{r-1}) = \sum \check{\lambda}_i \otimes f_i$ if and only if $u_{r-1}(\mathcal{U}_G^*(\lambda_i)) = f_i$ if and only if $\mathcal{U}_{G,r}^*(\lambda_i) = \ldots + f_i T^{p^{r-1}} + \ldots$. Hence, the assertion that $\Theta_{G,r}$ is homogeneous of degree p^{r-1} is equivalent to showing that the map $k[G] \to A$ defined by reading off the coefficient of

$$\mathcal{U}_{G,r}^* : k[G] \ \to \ A \otimes k[G] \ \to \ A \otimes k[\mathbb{G}_{a(r)}] \ \to \ A[T]/T^{p^r}$$

of the monomial $T^{p^{r-1}}$ is homogeneous of degree p^{r-1}.

The coordinate algebra $k[\mathbb{G}_{a(r)}] \simeq k[T]/T^{p^r}$ has a natural grading with T assigned degree 1. This grading corresponds to the monoidal action of $\mathbb{A}^1$ on $\mathbb{G}_{a(r)}$ by multiplication:

$$\mathbb{G}_{a(r)} \times \mathbb{A}^1 \xrightarrow{s \times a \mapsto sa} \mathbb{G}_{a(r)} \ .$$

We proceed to prove that this action is compatible with the action of $\mathbb{A}^1$ on $V_r(G)$ which defines the grading on A in the sense that the following diagram commutes:

(2.8.1)

$$\begin{tikzcd}
\mathbb{G}_{a(r)} \times \mathbb{A}^1 \times V_r(G) \arrow[r, "\mathrm{id} \times \mathrm{action}"] \arrow[d, "\mathrm{action} \times \mathrm{id}"] & \mathbb{G}_{a(r)} \times V_r(G) \arrow[r, "\mathcal{U}_{G,r}"] & G \times V_r(G) \arrow[d, "pr_G"] \\
\mathbb{G}_{a(r)} \times V_r(G) \arrow[r, "\mathcal{U}_{G,r}"] & G \times V_r(G) \arrow[r, "pr_G"] & G.
\end{tikzcd}$$

Commutativity of (2.8.1) is equivalent to the commutativity of the corresponding diagram of S-valued points for any choice of finitely generated commutative k-algebras S and element $a \in S$:

(2.8.2)

$$\begin{tikzcd}
\mathbb{G}_{a(r)}(S) \times V_r(G)(S) \arrow[r, "1 \times a"] \arrow[d, "a \times 1"] & \mathbb{G}_{a(r)}(S) \times V_r(G)(S) \arrow[r, "\mathcal{U}_{G,r}(S)"] & G(S) \times V_r(G)(S) \arrow[d, "pr_G"] \\
\mathbb{G}_{a(r)}(S) \times V_r(G)(S) \arrow[r, "\mathcal{U}_{G,r}(S)"] & G(S) \times V_r(G)(S) \arrow[r, "pr_G"] & G(S).
\end{tikzcd}$$

Choose an embedding of G into some $\mathrm{GL}_{N(r)}$. Using Lemma 2.7 and the naturality with respect to change of G of the action of $\mathbb{A}^1$ on $V_r(G)$, we can compare the diagram (2.8.2) for G and for $\mathrm{GL}_{N(r)}$. The injectivity of $G(S) \to \mathrm{GL}_{N(r)}(S)$ implies that it suffices to assume that $G = \mathrm{GL}_{N(r)}$. Let $s \in \mathbb{G}_{a(r)}(S)$, $\underline{\alpha} = (\alpha_0, \ldots, \alpha_{r-1}) \in V_r(\mathrm{GL}_N)(S)$. Then $a \circ \underline{\alpha} = (a\alpha_0, a^p\alpha_1, \ldots, a^{p^{r-1}}\alpha_{r-1})$, and

$\exp_{\underline{\alpha}}(s) = \exp(s\alpha_0)\exp(s^p\alpha_1)\dots\exp(s^{p^{r-1}}\alpha_{p-1}) \in \mathrm{GL}_{N(r)}(S)$. Thus, restricted to the point $(s,\underline{\alpha}) \in (\mathbb{G}_{a(r)} \times V_r(\mathrm{GL}_N))(S)$, (2.8.2) becomes

(2.8.3)
$$\begin{tikzcd}
(s,\underline{\alpha}) \arrow[r, "1\times a"] \arrow[d, "a\times 1"] & (s, a\circ\underline{\alpha}) \arrow[r, "\mathcal{U}_{G,r}"] & (\exp_{a\circ\underline{\alpha}}(s), a\circ\underline{\alpha}) \arrow[d] \\
(as,\underline{\alpha}) \arrow[r, "\mathcal{U}_{G,r}"] & (\exp_{\underline{\alpha}}(as), \underline{\alpha}) \arrow[r] & \exp_{\underline{\alpha}}(as).
\end{tikzcd}$$

Commutativity of (2.8.3) is implied by the equality $\exp_{a\circ\underline{\alpha}}(s) = \exp_{\underline{\alpha}}(as)$ which follows immediately by direct inspection of the formulas in [22, p.9]

Consequently, we have a commutative diagram on coordinate algebras corresponding to (2.8.1):

(2.8.4)
$$\begin{tikzcd}
k[\mathbb{G}_{a(r)}]\otimes k[t]\otimes A & k[\mathbb{G}_{a(r)}]\otimes A \arrow[l, "\mathrm{id}\otimes\mathrm{act}^*"'] & k[G]\otimes A \arrow[l, "\mathcal{U}^*_{G,r}"'] \\
k[\mathbb{G}_{a(r)}]\otimes A \arrow[u, "(-\otimes t)\otimes\mathrm{id}"'] & k[G]\otimes A \arrow[l, "\mathcal{U}^*_{G,r}"'] & k[G]. \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

The map $\mathrm{act}^* : A \longrightarrow k[t]\otimes A = k[\mathbb{A}^1]\otimes A$ is the map on coordinate algebras which corresponds to the grading on A. The left vertical map corresponds to the grading on $k[\mathbb{G}_{a(r)}] \simeq k[T]/T^{p^r}$ and is given explicitly by $T \mapsto T\otimes t$.

For $\lambda \in k[G]$, write $\mathcal{U}^*_G(\lambda\otimes 1) = \sum c_i T^i \otimes f_i \in k[\mathbb{G}_{a(r)}]\otimes A$. The composition of the lower horizontal and left vertical maps of (2.8.4) sends λ to $\sum c_i T^i \otimes t^i \otimes f_i$. On the other hand, the composition of the right vertical and upper horizontal maps of (2.8.4) sends λ to $\sum c_i T^i \otimes (\mathrm{act}^*(f_i))$. We conclude that

$$t^i \otimes f_i = \mathrm{act}^*(f_i),$$

so that f_i is homogeneous of degree i. □

As a corollary (of the proof of) Proposition 2.8, we see why for G infinitesimal of height $\leq r$ the homogeneous degree of $\Theta_{G,r} \in k[V_r(G)]\otimes kG$ is p^{r-1} whereas the homogeneous degree of $\Theta_{G,r+1} \in k[V_{r+1}(G)]\otimes kG$ is p^r.

Corollary 2.9. *Let G be an infinitesimal group of height $\leq r$. Then the map $i^* : k[V_{r+1}(G)] \to k[V_r(G)]$ of Proposition 2.4 is a graded isomorphism which divides degrees by p.*

Proof. Let $\pi^* : k[V_r(G)] \to k[V_{r+1}(G)]$ be the inverse of i^*. The commutativity of (2.8.1) implies that we may compute the effect on degree of π^* by identifying the effect on degree of the map $p^* : k[\mathbb{G}_{a(r)}] = k[[t]/t^{p^r} \to k[t]/t^{p^{r+1}} = k[\mathbb{G}_{a(r+1)}]$. Yet this map clearly multiplies degree by p. □

3. θ_v and local Jordan type

The purpose of this section is to exploit our universal p-nilpotent operator Θ_G to investigate the local Jordan type of a finite dimensional kG-module M. The local Jordan type of M gives much more detailed information about a kG-module M than the information which can be obtained from the support variety (or, rank variety) of M. In this section, we work through various examples, give an algorithm for computing local Jordan types, and understand the effect of Frobenius twists.

Moreover, we establish restrictions on the rank and dimension of kG-modules of constant Jordan type.

Definition 3.1. Let G be an infinitesimal group scheme and $v \in V(G)$. Let $k(v)$ denote the residue field of $V(G)$ at v, and let

$$\mu_v = k(v) \otimes_{k[V(G)]} \mathcal{U}_G : \mathbb{G}_{a(r),k(v)} \ \to \ G_{k(v)}$$

be the associated 1-parameter subgroup (for $r \geq \mathrm{ht(G)}$). We define the *local p-nilpotent operator at v, θ_v* , to be

$$\theta_v = k(v) \otimes_{k[V(G)]} \Theta_G = \ (\mu_{v*} \circ \epsilon)(u) \ \in \ k(v)G.$$

In the special case that $G = \mathrm{GL}_{n(r)}$ for some $n > 0$, we use the alternate notation $\theta_{\underline{\alpha}}$ for the local p-nilpotent operator at $\underline{\alpha} = (\alpha_0, \dots, \alpha_{r-1}) \in V(\mathrm{GL}_{n(r)}) \simeq N_p^{[r]}(gl_n)$:

$$\theta_{\underline{\alpha}} \ = \ (\exp_{\underline{\alpha},*} \circ \epsilon)(u) \ \in \ k(\underline{\alpha})\, \mathrm{GL}_{n(r)}, \tag{3.1.1}$$

where $k(\underline{\alpha})$ is the residue field of $\underline{\alpha} \in V(\mathrm{GL}_{n(r)})$.

Let K be a field. Then a finite dimensional $K[u]/u^p$-module M is a direct sum of cyclic modules of dimension ranging from 1 to p. We may thus write $M \simeq a_p[p] + \cdots + a_1[1]$, where $[i]$ is the cyclic $K[u]/u^p$-module $K[u]/u^i$ of dimension i. We refer to the p-tuple

$$\mathrm{JType}(M, u) \ = \ (a_p, \dots, a_1) \tag{3.1.2}$$

as the *Jordan type* of the $K[u]/u^p$-module M. We also refer to $\mathrm{JType}(M, u)$ as the Jordan type of the p-nilpotent operator u on M.

For simplicity, we introduce the following notation.

Definition 3.2. With notation as in Definition 3.1, we set

$$\mathrm{JType}(M, \theta_v) \ \equiv \ \mathrm{JType}((\mu_{v,*} \circ \epsilon)^*(M_{k(v)}), u).$$

We refer to this Jordan type as the local Jordan type of M at $v \in V(G)$.

The following proposition will enable us to make more concrete and explicit the local Jordan type of a kG-module M at a given 1-parameter subgroup of G.

Proposition 3.3. *Let $\underline{\alpha} = (\alpha_0, \dots, \alpha_{r-1}) \in V(\mathrm{GL}_{n(r)})$ be an r-tuple of p-nilpotent pair-wise commuting matrices. Let M be a $k\,\mathrm{GL}_{(r)}$-module of dimension N, and let $\rho : \mathrm{GL}_{m(r)} \to \mathrm{GL}_N$ be the associated structure map. The (i,j)-matrix entry of the action of the local p-nilpotent operator $\theta_{\underline{\alpha}} \in k(\underline{\alpha})\, \mathrm{GL}_{n(r)}$ of (3.1.1) on M equals the coefficient of $t^{p^{r-1}}$ of*

$$(\exp_{\underline{\alpha}})^*(\rho^* X_{ij}) \ \in \ k(\underline{\alpha})[\mathbb{G}_{a(r)}],$$

where $\{X_{ij}, 1 \leq i, j \leq N\}$ are the coefficient functions of GL_N.

Proof. Let $\langle m_i \rangle_{1 \leq i \leq N}$ be the basis of M corresponding to the structure map ρ. The structure of M as a comodule for $k[\mathrm{GL}_{n(r)}]$ is given by

$$M \to M \otimes k[\mathrm{GL}_{n(r)}], \quad m_j \mapsto \sum_i m_i \otimes \rho^* X_{ij},$$

and thus the comodule structure of $M_{k(\underline{\alpha})}$ for $k(\underline{\alpha})[\mathbb{G}_{a(r)}]$ is given by

$$M \to M \otimes k(\underline{\alpha})[\mathbb{G}_{a(r)}], \quad m_j \mapsto \sum_i m_i \otimes \exp^*_{\underline{\alpha}}(\rho^* X_{ij}).$$

The proposition follows from the fact that $u_{r-1} : k(\underline{\alpha})[\mathbb{G}_{a(r)}] \to k(\underline{\alpha})$ is given by reading off the coefficient of $t^{p^{r-1}} \in k(\underline{\alpha})[\mathbb{G}_{a(r)}]$. $\square$

Example 3.4. We investigate the local Jordan type of the various representations considered in Example 2.6.

(1) Consider the adjoint representation $M = \mathfrak{g}^{\mathrm{ad}}$ of a p-restricted Lie algebra $\mathfrak{g}$ and a 1-parameter subgroup

$$\mu_x : \mathbb{G}_{a(1),K} \ \to \ \underline{\mathfrak{g}}_K, \quad \text{inducing} \ \ K[u]/u^p \ \to \ \mathfrak{u}(\mathfrak{g}_K)$$

sending u to some p-nilpotent $X \in \mathfrak{g}_K$. The local Jordan type of $\mathfrak{g}^{\mathrm{ad}}$ at μ_x is simply the Jordan type of the endomorphism $\mathrm{ad}_x : \mathfrak{g}_K^{ad} \to \mathfrak{g}_K^{\mathrm{ad}}$,

$$\mathrm{JType}(\mathfrak{g}^{\mathrm{ad}}, \theta_x) \ = \ \mathrm{JType}(x).$$

(2) Let $M = k[\mathbb{G}_{a(r)}]/(u_0) \simeq k[u_1, \dots, u_{r-1}]/(u_1^p, \dots, u_{p-1}^p)$ be a cyclic $k\mathbb{G}_{a(r)} = k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{p-1}^p)$-module, and let $\mu_{\underline{a}} : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$ be a 1-parameter subgroup for some K-rational point $\underline{a}$ of $V(\mathbb{G}_{a(r)}) = \mathbb{A}^r$. Then

$$\mathrm{JType}(M, \theta_{\underline{a}}) = \begin{cases} p^{r-2}[p], & \exists i > 0, \ a_i \neq 0 \\ p^{r-1}[1], & \text{otherwise.} \end{cases}$$

(3) Let $G = \mathrm{GL}_{n(r)}$, and let V_n be the canonical n-dimensional rational representation of $\mathrm{GL}_{n(r)}$. We apply Proposition 3.3, observing that ρ for V_n is simply the natural inclusion $\mathrm{GL}_{n(r)} \subset \mathrm{GL}_n$. Since

$$\exp^*_{\underline{\alpha}}(X_{i,j}) = \sum_{\ell=0}^{p^r-1} [\beta_\ell]_{i,j} t^\ell, \tag{3.4.1}$$

where β_ℓ are matrices determined by α_i as in Proposition 1.6, we conclude

$$\mathrm{JType}(V_n, \theta_{\underline{\alpha}}) \ = \ \mathrm{JType}(\alpha_{r-1}).$$

Specializing to $r = 2$,

$$\mathrm{JType}(V_n, \theta_{(\alpha_0, \alpha_1)}) \ = \ \alpha_1.$$

(4) "Specializing" to $G = \mathrm{SL}_{2(2)}$, consider $\underline{\alpha} = (\begin{bmatrix} c_0 & a_0 \\ b_0 & -c_0 \end{bmatrix}, \begin{bmatrix} c_1 & a_1 \\ b_1 & -c_1 \end{bmatrix})$. Then $\mathrm{JType}(V_2, \theta_{\underline{\alpha}})$ equals the Jordan type of the matrix $\begin{bmatrix} c_1 & a_1 \\ b_1 & -c_1 \end{bmatrix}$.

We extend Example 3.4(3) by considering tensor powers $V_n^{\otimes d}$ of the canonical rational representation of GL_n restricted to $\mathrm{GL}_{n(2)}$. In this example, the role of both entries of the pair $\underline{\alpha} = (\alpha_0, \alpha_1)$ is non-trivial.

Example 3.5. Consider the $N = n^d$-dimensional rational GL_n-module $M = V_n^{\otimes d}$ where V_n is the canonical n-dimensional rational GL_n-module. Let $\rho : \mathrm{GL}_{n(r)} \to \mathrm{GL}_N$ be the representation of M restricted to $\mathrm{GL}_{n(r)}$. A basis of M is $\{e_{i_1} \otimes \cdots \otimes e_{i_d};\ 1 \leq i_j \leq n\}$, where $\{e_i; 1 \leq i \leq n\}$ is a basis for V_n. Let $\{X_{i_1,j_1;\dots,i_d,j_d},\ 1 \leq i_t, j_t \leq n\}$ denote the matrix coefficients on GL_N, and let $\{Y_{s,t},\ 1 \leq s,t \leq n\}$ denote the matrix coefficients of GL_n.

Then $\rho^* : k[\mathrm{GL}_N] \to k[\mathrm{GL}_{n(r)}]$ is given by

$$X_{i_1,j_1;\dots;i_d,j_d} \mapsto \; Y_{i_1,j_1} \cdots Y_{i_d,j_d}.$$

Thus,

$$(\exp_{\underline{\alpha}})^*(\rho^*(X_{i_1,j_1;\dots,i_d,j_d})) \;=\; (\exp_{\underline{\alpha}})^*(Y_{i_1,j_1}) \cdots (\exp_{\underline{\alpha}})^*(Y_{i_d,j_d}).$$

Now, specialize to $r = 2$ so that we can make this more explicit. Then the coefficient of t^p of $(\exp_{(\alpha_0,\alpha_1)})^*(\rho^*(X_{i_1,j_1;\dots,i_d,j_d}))$ is

$$\sum_{k=1}^{d}(\alpha_1)_{i_k,j_k} + \sum_{\substack{0 \le f_k < p \\ f_1+\cdots+f_d=p}} \frac{1}{f_1!} \cdots \frac{1}{f_d!}((\alpha_0)^{f_1})_{i_1,j_1} \cdots ((\alpha_0)^{f_d})_{i_d,j_d}. \tag{3.5.1}$$

This gives the action of $\theta_{(\alpha_0,\alpha_1)}$ on M.

To simplify matters even further, consider the special case $(\alpha_0)^2 = 0$. For $1 \le d < p$, $\theta_{(\alpha_0,\alpha_1)}$ on M is given by the $N \times N$-matrix

$$\big(i_1, j_1; \dots ; i_d, j_d\big) \mapsto \big(\sum_{k=1}^{d}(\alpha_1)_{i_k,j_k}\big).$$

For $d = p$, the action of $\theta_{(\alpha_0,\alpha_1)}$ on M is given by the $N \times N$-matrix

$$\big(i_1, j_1; \dots ; i_p, j_p\big) \mapsto \big(\sum_{k=1}^{p}(\alpha_1)_{i_k,j_k} + (\alpha_0)_{i_1,j_1} \cdots (\alpha_0)_{i_p,j_p}\big).$$

An analogous calculation applies to the the d-fold symmetric product $S^d(V_n)$ and d-fold exterior product $\Lambda^d(V_n)$ of the canonical n-dimensional rational GL_n-module V_n.

The proof of Proposition 3.3 applies equally well to prove the following straightforward generalization, which one may view as an algorithmic method of computing the "local Jordan type" of a kG-module M of dimension N. The required input is an explicit description of the map on coordinate algebras ρ^* given by $\rho : G \to \mathrm{GL}_N$ determining the kG-module M.

Theorem 3.6. *Let G be an infinitesimal group scheme of height $\le r$, and let $\rho : G \to \mathrm{GL}_N$ be a representation of G on a vector space M of dimension N. Consider some $v \in V(G)$, and let $\mu_v : \mathbb{G}_{a(r),k(v)} \to G_{k(v)}$ be the corresponding 1-parameter subgroup of height r. Then the (i, j)-matrix entry of the action of $\theta_v \in k(v)G$ on M equals the coefficient of $t^{p^{r-1}}$ of*

$$(\mu_v)^*(\rho^* X_{ij}) \;\in\; k(v)[\mathbb{G}_{a(r)}],$$

where $\{X_{ij}, 1 \le i, j \le N\}$ are the coefficient functions of GL_N.

As a simple corollary of Theorem 3.6, we give a criterion for the local Jordan type of the kG-module M to be trivial (i.e., equal to $(\dim M)[1]$) at a 1-parameter subgroup $\mu_v,\ v \in V(G)$.

Corollary 3.7. *With the hypotheses and notation of Theorem 3.6,*

$$\mathrm{JType}(M, \theta_v) \;=\; \mathrm{JType}(\mu_{v*} \circ \epsilon)^*(M_{k(v)}), u) \;=\; (\dim M)[1]$$

if $\deg\,(\rho \circ \mu_v)^*(X_{ij}) < p^{r-1}$ *for all* $1 \le i, j \le N$.

One means of constructing kG-modules is by applying Frobenius twists to known kG-modules. Our next objective is to establish (in Proposition 3.9) a simple relationship between the p-nilpotent operator $\theta_{\underline{\alpha}}$ on a $k\,\mathrm{GL}_{n(r)}$-module M and $\theta_{\underline{\alpha}}$ on the s-th Frobenius twist $M^{(s)}$ of M for any $0 \neq v \in V(\mathrm{GL}_{n(r)})$.

Before formulating this relationship, we make explicit the definition of the Frobenius map for an arbitrary affine group scheme over k. Let G be an affine group scheme over k and define for any $s > 0$ the s^{th} Frobenius map $F^s : G \to G^{(s)}$ given by the k-linear algebra homomorphism

$$F^{s*} : k[G^{(s)}] = k \otimes_{p^s} k[G] \;\to\; k[G], \quad a \otimes f \mapsto a \cdot f^{p^s}, \tag{3.7.1}$$

where $k \otimes_{p^s} k[G]$ is the base change of $k[G]$ along the p^s-power map $k \to k$ (an isomorphism only for k perfect). If G is defined over $\mathbb{F}_{p^s}$ (for example, if $G = \mathrm{GL}_n$)), then we have a natural isomorphism

$$k[G] = k \otimes_{\mathbb{F}_{p^s}} \mathbb{F}_{p^s}[G] \xrightarrow{\sim} k \otimes_{p^s} k \otimes_{\mathbb{F}_{p^s}} \mathbb{F}_{p^s}[G] = k[G^{(s)}]$$

so that F^s can be viewed as a self-map of G.

Definition 3.8. If M is a kG-module, then the s^{th} Frobenius twist $M^{(s)}$ of M is the k-vector space $k \otimes_{p^s} M$. By naturality, $M^{(s)}$ inherits a $kG^{(s)}$-module structure. We view $M^{(s)}$ as a kG-module via the map $F^s_* : kG \to kG^{(s)}$ dual to (3.7.1).

To be more explicit, suppose the N-dimensional kG-module M is given by $\rho : G \to \mathrm{GL}_N$ (so that $M = \rho^*(V_N)$, where V_N is the canonical N-dimensional GL_N-module) and assume that G is defined over $\mathbb{F}_{p^s}$. Let $\mu_v : \mathbb{G}_{a(r),K} \to G_K$ be a 1-parameter subgroup, corresponding to some $v \in V(G)$. Then the identification of $M^{(s)}$ with $(\rho \circ F^s)^*(V_N)$ implies that

$$\mathrm{JType}(M^{(s)}, \theta_v) \;=\; \mathrm{JType}(M, \theta_{F^s(v)}) \tag{3.8.1}$$

where $\theta_{F^s(v)} = ((F^s \circ \mu_v)_* \circ \epsilon)(u)$.

Let $G = \mathrm{GL}_{n(r)}$, and let R be a finitely generated commutative k-algebra. The Frobenius self-map is given explicitly on the R-valued of $\mathrm{GL}_{n(r)}$ by the formula

$$F : \alpha \mapsto \phi(\alpha),$$

where ϕ applied to $\alpha \in M_n(R)$ raises each entry of α to the p-th power. For t in $\mathbb{G}_{a(r)}(R)$, we compute

$$(F \circ \exp_{(\alpha_0,\dots,\alpha_{r-1})})(t) = F(\exp(t\alpha_0)\exp(t^p\alpha_1)\dots\exp(t^{p^{r-1}}\alpha_{r-1})) =$$

$$\exp(t^p\phi(\alpha_0))\exp(t^{p^2}\phi(\alpha_1))\dots\exp(t^{p^{r-1}}\phi(\alpha_{r-2})) = \exp_{(0,\phi(\alpha_0),\dots,\phi(\alpha_{r-2}))}(t).$$

Iterating s times, we obtain the following formula for $G = \mathrm{GL}_{n(r)}$:

$$F^s \circ \exp_{(\alpha_0,\dots,\alpha_{r-1})} \;=\; \exp_{(0,0,\dots,0,\phi^s(\alpha_0),\dots,\phi^s(\alpha_{r-1-s})} \tag{3.8.2}$$

where the first non-zero entry on the right happens at the $(s+1)$-st place.

In another special case of $G = \mathbb{G}_{a(r)}$ the Frobenius map $F : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$ is given by raising an element $a \in \mathbb{G}_{a(r)}(R)$ to the p-th power. Let $\underline{a} = (a_0, \dots, a_{r-1})$ be a point in $V(\mathbb{G}_{a(r)}) \simeq \mathbb{A}^r$, and let $\mu_{\underline{a}} : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$ be the corresponding 1-parameter subgroup. For $t \in \mathbb{G}_{a(r)}(R)$, we have $\mu(t) = a_0 + a_1 t + \dots + a_{r-1}t^{p-1}$ (see [22, §1]). The following formula is now immediate:

$$F^s \circ \mu_{(a_0,\dots,a_{r-1})} \;=\; \mu_{(0,\dots,0,a_0^{p^s},\dots,a_{r-1-s}^{p^s})}. \tag{3.8.3}$$

Combining (3.8.1) and (3.8.2), we conclude the following proposition.

Proposition 3.9. *Let M be a finite dimensional $k\,\mathrm{GL}_{n(r)}$-module and let $\underline{\alpha} = (\alpha_0, \ldots, \alpha_{r-1})$ be a point in $V(\mathrm{GL}_{n(r)})$. Then*

$$\mathrm{JType}(M^{(s)}, \theta_{\underline{\alpha}}) \;=\; \mathrm{JType}(M, \theta_{F^s \circ \underline{\alpha}}),$$

where $F^s \circ \underline{\alpha} = (0, \ldots, 0, \phi^s(\alpha_1), \ldots, \phi^s(\alpha_{r-1-s}))$.

Similarly, if M be a finite dimensional $k\mathbb{G}_{a(r)}$-module, and $\underline{a} = (a_0, \ldots, a_{r-1})$ be a point in $V(\mathbb{G}_{a(r)}) \simeq \mathbb{A}^r$. Then

$$\mathrm{JType}(M^{(s)}, \theta_{\underline{a}}) \;=\; \mathrm{JType}(M, \theta_{F^s \circ \underline{a}}),$$

where $F^s \circ \underline{a} = (0, \ldots, 0, a_0^{p^s}, \ldots, a_{r-1-s}^{p^s})$.

Proposition 3.9 has the following immediate corollary.

Corollary 3.10. *Let M be a finite dimensional $k\,\mathrm{GL}_{n(r)}$-module. Then $\underline{\alpha} = (\alpha_0, \ldots, \alpha_{r-1}) \in V(\mathrm{GL}_{n(r)})$ lies in the rank variety $V(\mathrm{GL}_{n(r)})_{M^{(s)}}$ (as defined in (1.14)) provided that $\alpha_0 = \cdots = \alpha_{r-1-s} = 0$.*

The following definition introduces interesting classes of kG-modules which have special local behavior.

Definition 3.11. Let G be an infinitesimal group scheme and j a positive integer less than p. A finite dimensional kG-module M is said to be of constant j-rank if and only if

$$\mathrm{rk}(M, \theta_v^j) \;\equiv\; \mathrm{rk}\{\theta_v^j : M_{k(v)} \to M_{k(v)}\}$$

is independent of $v \in V(G) - \{0\}$, where θ_v is the local p-nilpotent operator at v as introduced in Definition 3.1.

M is said to be of constant Jordan type if and only if it is of constant j-rank for all j, $1 \leq j < p$. M is said to be of constant rank if it is of constant 1-rank.

As we see in the following example, one can have rational GL_n-modules of constant Jordan type when restricted to $\mathrm{GL}_{n(r)}$ of arbitrarily high degree d. This should be contrasted with Corollary 3.16.

Example 3.12. Consider the rational GL_n-module $M = \det^{\otimes d}$, the d^{th} power of the determinant representation. This is a polynomial representation of degree n^d. The restriction of M to any Frobenius kernel $\mathrm{GL}_{n(r)}$ has trivial constant Jordan type, for the further restriction of M to any abelian unipotent subgroup of GL_n is trivial.

One method of constructing kG-modules of constant Jordan type is to start with some kG-module M of constant Jordan type (for example, take M to be the trivial kG-module k) and consider the n-th "syzygy" of M, $\Omega^n(M)$, $n \in \mathbb{Z}$. Recall that the syzygies $\Omega^n(M)$ of a kG-module are defined in terms of the minimal projective resolution of M (see, for example, [2]).

Example 3.13. Let $\mathfrak{G}$ be a reduced, irreducible group scheme and $G = \mathfrak{G}_{(r)}$. Let $k \to k[G] \to \cdots \to k[G^{\times s}] \to \cdots$ be the cobar resolution of k by free $k[G]$-modules, so that the dual $\cdots \to kG^{\times s} \cdots \to kG \to k$ is a resolution of k by free kG-modules. Since $k[G]$ is self-injective, the cobar resolution is also a resolution by injective kG-modules. Since each $k[G^{\times s}]$ is a rational $\mathfrak{G}$-module and each map of the cobar resolution is a map of $\mathfrak{G}$-modules, we conclude that the Heller shifts $\Omega^i(k),\ i \in \mathbb{Z}$ are all rational $\mathfrak{G}$-modules. On the other hand, each $\Omega^i(k)$ has constant Jordan type as a kG-module, of Jordan type of the form $m[p]+[1]$ if i is even and $m[p]+[p-1]$ if i is odd.

We shall see below that kG-modules of constant j-rank lead to interesting constructions of vector bundles (see Theorem 5.1). We conclude this section by establishing two constraints, Propositions 3.15 and 3.18, on kG-modules to be modules of constant rank.

We first need the following elementary lemma.

Lemma 3.14. *Let M be a $\mathbb{G}_{a(r)}$-module such that the local Jordan type at every $v \in V(\mathbb{G}_{a(r)})$ is trivial. Then M is trivial.*

Proof. The action of $\mathbb{G}_{a(r)}$ on M is given by the action of r commuting p-nilpotent operators $\tilde{u}_i,\ 0 \leq i < p$ on M. Moreover

$$\mathrm{JType}(M, \theta_{\underline{a}}) \ = \ \mathrm{JType}(a_{r-1}\tilde{u}_0 + a_{r-2}^p \tilde{u}_1 + \cdots + a_0^{p^{r-1}} \tilde{u}_{r-1})$$

(see Example 2.5(2)). Thus, if the local Jordan type of M is trivial at each $\underline{a} = (0, \ldots, 1, 0, \ldots, 0)$, then each $\tilde{u}_i$ must act trivially on M and M is therefore a trivial $\mathbb{G}_{a(r)}$-module. □

Proposition 3.15. *Let M be a non-trivial rational $\mathbb{G}_a$-module given by $\rho : \mathbb{G}_a \to \mathrm{GL}_N$. Let D be an upper bound for the degrees of the polynomials $\rho^*(X_{ij}) \in k[\mathbb{G}_a]$ where $\{X_{ij}\}$ are the standard polynomial generators of $k[\mathrm{GL}_N]$. Then M is not a $k\mathbb{G}_{a(r)}$-module of constant rank provided that $r > log_p D + 1$.*

Proof. The condition $r > log_p D$ implies that M is not r-twisted (i.e., of the form $N^{(r)}$). Since M is is not r-twisted, it is necessarily non-trivial as a $\mathbb{G}_{a(r)}$-module. Lemma 3.14 implies that the local Jordan type of M at some 1-parameter subgroup $\mu_v : \mathbb{G}_{a(r),k(v)} \to G_{k(v)}$ is non-trivial. On the other hand, Corollary 3.7 implies that the Jordan type of M at the identity 1-parameter subgroup $\mathrm{id} : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$ is trivial provided that $r - 1 > \log_p D$. □

The preceding theorem enables us to conclude that various rational modules M for algebraic groups $\mathfrak{G}$ are not of constant Jordan type when restricted to $\mathfrak{G}_{(r)}$ for r sufficiently large. Namely, we apply Proposition 3.15 to the restriction of M to some 1-parameter subgroup $\mathbb{G}_a \to \mathfrak{G}$.

Because SL_n is generated by its 1-parameter subgroups, we obtain the following corollary (which should be contrasted with Example 3.12).

Corollary 3.16. *Let M be a non-trivial polynomial SL_n-module of degree D. If $r > \log_p D + 1$, then M is not a $k\,\mathrm{SL}_{n(r)}$-module of constant rank.*

The following lemma, which is a straightforward application of the Generalized Principal Ideal Theorem (see [9, 10.9]), shows that the dimension of a non-trivial module of constant rank of $\mathbb{G}_{a(r)}$ cannot be "too small" compared to r.

Lemma 3.17. *Let M be a finite dimensional $\mathbb{G}_{a(r)}$-module. If M is a non-trivial $\mathbb{G}_{a(r)}$-module of constant rank, then the following inequality holds:*

$$\dim_k M \geq \sqrt{r} \tag{3.17.1}$$

Proof. By extending scalars if necessary we may assume that k is algebraically closed. Let $m = \dim_k M$. Let $k\mathbb{G}_{a(r)} = k[u_0, \dots .u_{r-1}]/(u_0^p, \dots, u_{r-1}^p)$, let $K = k(s_0, \dots, s_{r-1})$ where s_i are independent variables, and let $\alpha_K : K[t]/t^p \to K\mathbb{G}_{a(r)}$ be a generic π-point given by $\alpha_K(t) = s_0u_0 + \cdots + s_{r-1}u_{r-1}$. Choose a k-linear basis of M, and let $A(s_0, \dots, s_{r-1})$ be a nilpotent matrix in $M_m(k[s_0, \dots, s_{r-1}])$ representing the action of $\alpha_K(t)$ on M_K. Let $I_n(A(s_0, \dots, s_{r-1}))$ denote the ideal generated by all $n \times n$ minors of $A(s_0, \dots, s_{r-1})$. By [9, 10.9], the codimension of any minimal prime over $I_n(A(s_0, \dots, s_{r-1}))$ is at most $(m - n + 1)^2$.

Assume that (3.17.1) does not hold, that is, $m < \sqrt{r}$. Hence, $(m - n + 1)^2 < r$ for any $1 \leq n \leq m$. The variety of $I_n(A(s_0, \dots, s_{r-1}))$ is a subvariety inside $\operatorname{Spec} k[s_0, \dots, s_{r-1}] \simeq \mathbb{A}^r$ which has dimension r. Since the codimension of the variety of $I_n(A(s_0, \dots, s_{r-1}))$ is at most $(m-n+1)^2$, we conlude that the dimension is at least $r-(m-n+1)^2 \geq 1$. Hence, the minors of dimension $n \times n$ have a common non-trivial zero. Taking $n = 1$, we conclude that $A(b_0, \dots, b_{r-1})$ is a zero matrix for some non-zero specialization $b_0, \dots, b_{r-1}$ of $s_0, \dots, s_{r-1}$. Consequently, M is trivial at the π-point of $\mathbb{G}_{a(r)}$ corresponding to $b_0, \dots, b_{r-1}$. Since M is non-trivial, Lemma 3.14 implies that M is not a module of constant rank. □

As an immediate corollary, we provide an additional necessary condition for a $k\mathfrak{G}_{(r)}$-module to have constant rank.

Proposition 3.18. *Let $\mathfrak{G}$ be a (reduced) affine algebraic group and M be a rational representation of $\mathfrak{G}$. Assume that $\mathfrak{G}$ admits a 1-parameter subgroup $\mu : \mathbb{G}_{a(r)} \to \mathfrak{G}$ such that $\mu^*(M)$ is a non-trivial $k\mathbb{G}_{a(r)}$-module. If $r \geq (\dim M)^2 + 1$, then M is not a $k\mathfrak{G}_{(r)}$-module of constant rank.*

4. π-points and $\mathbb{P}(\mathrm{G})$

In a series of earlier papers, we have considered π-points for a finite group scheme G (as recalled in Definition 4.1) and investigated finite dimensional kG-modules M using the "Jordan type of M" at various π-points. In particular, in [15], we verified that this Jordan type is independent of the equivalence class of the π-point provided that either the π-point is generic or the Jordan type of M at some representative of the equivalence class is maximal.

As we recall below, whenever G is an infinitesimal group scheme, then the π-point space $\Pi(G)$ of equivalence classes of π-points is essentially the projectivization of $V(G)$. The purpose of the first half of this section is to relate the discussion of the previous section concerning the local Jordan type of a finite kG-module to our earlier work formulated in terms of π-points for general finite group schemes.

One special aspect of an infinitesimal group scheme G is that equivalence classes of π-points of G have canonical (up to scalar multiple) representatives.

Unless otherwise specified (as in Definition 4.1 immediately below), G will denote an infinitesimal group scheme over k, and $V(G)$ will denote $V_r(G)$ for some $r \geq \mathrm{ht}(\mathrm{G})$. Throughout this section we assume that $\dim V(G) \geq 1$, and work with $\mathbb{P}(\mathrm{G}) = \operatorname{Proj} V(G)$.

Definition 4.1. (see [14]) Let G be a finite group scheme.

(1) A π-point of G is a (left) flat map of K-algebras $\alpha_K : K[t]/t^p \to KG$ for some field extension K/k with the property that there exists a unipotent abelian closed subgroup scheme $i : C_K \subset G_K$ defined over K such that α_K factors through $i_* : KC_K \to KG_K = KG$.

(2) If $\beta_L : L[t]/t^p \to LG$ is another π-point of G, then α_K is said to be a *specialization* of β_L , written $\beta_L \downarrow \alpha_K$, provided that for any finite dimensional kG-module M, $\alpha_K^*(M_K)$ being free as $K[t]/t^p$-module implies that $\beta_L^*(M_L)$ is free as $L[t]/t^p$-module.
(3) Two π-points $\alpha_K : K[t]/t^p \to KG$, $\beta_L : L[t]/t^p \to LG$ are said to be *equivalent*, written $\alpha_K \sim \beta_L$, if $\alpha_K \downarrow \beta_L$ and $\beta_L \downarrow \alpha_K$.
(4) A π-point of G, $\alpha_K : K[t]/t^p \to KG$, is said to be *generic* if there does not exist another π-point $\beta_L : L[t]/t^p \to LG$ which specializes to α_K but is not equivalent to α_K.
(5) If M is a finite dimensional kG-module and $\alpha_K : K[t]/t^p \to KG$ a π-point of G, then the Jordan type of M at α_K is by definition the Jordan type of $\alpha_K^*(M_K)$ as $K[t]/t^p$-module.

Because the group algebra of a finite group scheme is always faithfully flat over the group algebra of a subgroup scheme (see [24, 14.1]), the condition on a flat map $\alpha_K : K[t]/t^p \to KG$ is equivalent to the existence of a factorization $i_* \circ \alpha'_K$ with $\alpha'_K : K[t]/t^p \to KC_K$ flat.

Definition 4.2. Let G be an infinitesimal scheme, and let $v \in V(G)$ be the point associated to the 1-parameter subgroup $\mu_v : \mathbb{G}_{a(r),k(v)} \to G_{k(v)}$. Then the π-point of G associated to v is

$$\mu_{v,*} \circ \epsilon : k(v)[t]/t^p \ \to \ k(v)G.$$

The following theorem is a complement to Theorem 1.15, revealing that spaces of (equivalence) classes of π-points are very closely related to (cohomological) support varieties.

Theorem 4.3. *(*[14, 7.5]*) Let G be an finite group scheme. Then the set of equivalence classes of π-points can be given a scheme structure, denoted $\Pi(G)$, which is defined in terms of the representation theory of G. Moreover, there is an isomorphism of schemes*

$$\mathrm{Proj}\ \mathrm{H}^\bullet(G,k) \ \simeq \ \Pi(G).$$

If G is an infinitesimal group scheme so that $\mathrm{H}^\bullet(G,k)$ is related to $k[V(G)]$ as in Theorem 1.15, then the resulting homeomorphism

$$\mathbb{P}(G) \ \xrightarrow{\sim} \ \Pi(G)_{\mathrm{red}} \tag{4.3.1}$$

is given on points by sending $x \in \mathbb{P}(G)$ to the equivalence class of the π-point $\mu_{v,*} \circ \epsilon$ for any $v \in V(G)\backslash\{0\}$ projecting to x. In particular, equivalence classes of generic π-points of G are represented by $(\mu_{v,*} \circ \epsilon)$ as $v \in V(G)$ runs through the (scheme-theoretic) generic points of $V(G)$.

Furthermore, for any finite dimensional kG-module M, (4.3.1) restricts to a homeomorphism of reduced, closed subvarieties

$$\mathbb{P}(G)_M \ \simeq \ \Pi(G)_M,$$

where $\mathbb{P}(G)_M = \mathrm{Proj}\, V(G)_M$, and $\Pi(G)_M$ consists of those equivalence classes of π-points α_K of G such that $\alpha_K^*(M_K)$ is not free (as a $K[u]/u^p$-module).

Generic π-points are particularly important when developing invariants of representations. The following corollary of Theorem 4.3 gives an explicit set of representatives of equivalence classes of generic π-points of G.

Proposition 4.4. *Let G be an infinitesimal group scheme with universal 1-parameter subgroup $\mathcal{U}_G : \mathbb{G}_{a(r),k[V(G)]} \longrightarrow G_{k[V(G)]}$. For each minimal prime ideal $\mathcal{P}_i$ of $k[V(G)]$, let K_i denote the field of fractions of $k[V(G)]/\mathcal{P}_i$. Then the compositions*

$$K_i \otimes_{k[V(G)]} \mathcal{U}_{G,*} \circ \epsilon : K_i[u]/u^p \to K_i G$$

(sending u to θ_{K_i}) are non-equivalent representatives of the equivalence classes of generic π-points of G.

Let G be an infinitesimal group scheme of height $\leq r$ and recall that $\Theta_{G,r} \in k[V_r(G)]$ is homogeneous of degree p^{r-1}.

Definition 4.5. Let G be an infinitesimal group scheme of height $\leq r$ and let M be a finite dimensional kG-module. Then we denote by

$$\widetilde{\Theta}_G : \mathcal{M} \equiv \mathcal{O}_{\mathbb{P}(\mathrm{G})} \otimes M \ \to \ \mathcal{O}_{\mathbb{P}(\mathrm{G})}(p^{r-1}) \otimes M \equiv \mathcal{M}(p^{r-1}) \tag{4.5.1}$$

the associated homomorphism of (locally free) coherent $\mathcal{O}_{\mathbb{P}(\mathrm{G})}$-modules determined by the action of $\Theta_{G,r} \in k[V_r(G)] \otimes kG$.

For any point $x \in \mathbb{P}(\mathrm{G})$, we use the notation

$$M_{k(x)} = k(x) \otimes_{\mathcal{O}_{\mathbb{P}(G)}} \mathcal{M}$$

for the fiber of the coherent sheaf $\mathcal{M}$ at x. Here, we have identified $k(x)$ with the residue field of the stalk $\mathcal{O}_{\mathbb{P}(\mathrm{G}),x}$.

Proposition 4.6. *Let G be an infinitesimal group scheme of height $\leq r$, and let M be a finite dimensional kG-module. For any $v,\ v' \in V(G)$ projecting to the same $x \in \mathbb{P}(\mathrm{G})$, we have*

$$\mathrm{Im}\{\theta_v : M_{k(v)} \to M_{k(v)}\} \ \simeq \ \mathrm{Im}\{\theta_{v'} : M_{k(v')} \to M_{k(v')}\}$$

and similarly for kernels.

Proof. This is essentially proved in [23, 6.1]. □

In the next section, we shall be particularly interested in kernels and images of $\widetilde{\Theta}_G$. The following proposition relates the local p-nilpotent operator θ_v on M at the point $v \in V(G)$ to the fiber of the action of $\widetilde{\Theta}_G$ on the coherent sheaf $\mathcal{M}$.

Proposition 4.7. *Let G be an infinitesimal group scheme of height $\leq r$, let M be a finite dimensional kG-module, and let $s \in \Gamma(\mathbb{P}(\mathrm{G}), \mathcal{O}_{\mathbb{P}(\mathrm{G})}(p^{r-1}))$ be a non-zero global section with zero locus $Z(s) \subset \mathbb{P}(\mathrm{G})$. Set $U = \mathbb{P}(\mathrm{G}) \backslash Z(s)$. Then there is a well defined endomorphism (depending upon s)*

$$\widetilde{\Theta}_G/s : \mathcal{M}_{|U} \ \to \ \mathcal{M}_{|U}.$$

Moreover, the image and kernel of the induced map $\theta_x/s : M_{k(x)} \to M_{k(x)}$ on fibers at $x \in U \subset \mathbb{P}(\mathrm{G})$ is independent of s and satisfies

$$\mathrm{Im}\{\theta_x/s : M_{k(x)} \to M_{k(x)}\} \simeq \mathrm{Im}\{\theta_v : M_{k(v)} \to M_{k(v)}\}$$

and

$$\mathrm{Ker}\{\theta_x/s : M_{k(x)} \to M_{k(x)}\} \simeq \mathrm{Ker}\{\theta_v : M_{k(v)} \to M_{k(v)}\}$$

for any $v \in V(G)\backslash\{0\}$ that projects onto x.

Proof. Let X denote $\mathbb{P}(\mathrm{G})$ and let $1/s \in \mathcal{O}_X(-p^{r-1})(U)$ satisfy

$$1/s \otimes s = 1 \ \in \mathcal{O}_X(-p^{r-1})(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(p^{r-1})(U) \ \simeq \ \mathcal{O}_X(U).$$

Then we define

$$\widetilde{\Theta}_G/s \equiv \ 1/s \otimes (\widetilde{\Theta}_G)_{|U} : \mathcal{M}_{|U} \to \mathcal{O}_X(-p^{r-1}))_{|U} \otimes \mathcal{M}(p^{r-1})_{|U} \ \simeq \ \mathcal{M}_{|U}.$$

The second statement is essentially proved in [23, 6.1]. □

Remark 4.8. For a finite group G, there is no natural choice of π-point representing a typical equivalence class $x \in \Pi(G) \simeq \operatorname{Proj} \mathrm{H}^\bullet(G,k)$ of π-points. As seen in elementary examples [15, 2.3], the Jordan type of a kG-module M typically can be different for two equivalent π-points representing the same point $x \in \Pi(G)$.

Remark 4.9. Proposition 4.7 immediately generalizes to $\widetilde{\Theta}_G^j$ for any $1 \leq j \leq p-1$. It implies the following isomorphisms for any $x \in X = \mathbb{P}(\mathrm{G})$, $v \in V(G)$ projecting onto x, and a global section s of $\mathcal{O}_X(jp^{r-1})$ such that $s(x) \neq 0$:

$$\operatorname{Im}\{(\theta_x/s)^j : M_{k(x)} \to M_{k(x)}\} \simeq \operatorname{Im}\{\theta_v^j : M_{k(v)} \to M_{k(v)}\} \simeq$$
$$\operatorname{Im}\{k(x) \otimes_{\mathcal{O}_X} \widetilde{\Theta}_G^j : k(x) \otimes_{\mathcal{O}_X} \mathcal{M} \to k(x) \otimes_{\mathcal{O}_X} \mathcal{M}(jp^{r-1})\},$$

and similarly for kernels.

In what follows, we shall use the following abbreviations:

$$\begin{aligned}
\operatorname{Im}\{\widetilde{\Theta}_G^j, \mathcal{M}\} \ &\equiv \ \operatorname{Im}\{\widetilde{\Theta}_G^j : \mathcal{M} \to \mathcal{M}(jp^{r-1})\},\\
\operatorname{Im}\{\theta_x^j, M_{k(x)}\} \ &\equiv \ \operatorname{Im}\{(\theta_x/s)^j : M_{k(x)} \to M_{k(x)}\},\\
\operatorname{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\} \ &\equiv \ \operatorname{Ker}\{\widetilde{\Theta}_G^j : \mathcal{M} \to \mathcal{M}(jp^{r-1})\},\\
\operatorname{Ker}\{\theta_x^j, M_{k(x)}\} \ &\equiv \ \operatorname{Ker}\{(\theta_x/s)^j : M_{k(x)} \to M_{k(x)}\}.
\end{aligned}$$

We shall verify in Theorem 4.12 that a necessary and sufficient condition on a finite dimensional kG-module M for $\operatorname{Im}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$ (and thus $\operatorname{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$) to be an algebraic vector bundle on X is that M be a module of constant j-type.

The following Proposition can be found as an Exercise in Hatshorne [17, 5 ex.5.8]

Proposition 4.10. *Let X be a reduced scheme and $\widetilde{M}$ a coherent $\mathcal{O}_X$-module. Then $\widetilde{M}$ is locally free if and only if* $\dim_{k(x)}(k(x) \otimes_{\mathcal{O}_X} \widetilde{M})$ *depends only upon the connected component of x in $\pi_0(X)$.*

Proof. Assume that the function $x \mapsto \dim_{k(x)}(k(x) \otimes_{\mathcal{O}_X} \widetilde{M})$ is constant on a connected component of X. To prove that M is locally free it suffices to assume that X is local so that $X = \operatorname{Spec} R$ for some reduced local commutative ring, and that M is a finite R-module with the property that $\dim_{k(p)}(k(p) \otimes_R M)$ is independent of the prime $p \subset R$. To prove that M is free, we choose some surjective R-module homomorphism $g : Q \to M$ from a free R-module $Q \simeq R^n$ with the property that $\overline{g} : R/\mathfrak{m} \otimes_R Q \to R/\mathfrak{m} \otimes_R M$ is an isomorphism where $\mathfrak{m} \subset R$ is the maximal ideal. Then g is surjective by Nakayama's lemma. By assumption, g induces an isomorphism after specialization to any prime $\mathfrak{p} \subset R$: $Q \otimes_{R_\mathfrak{p}} k(\mathfrak{p}) \simeq M \otimes_{R_\mathfrak{p}} k(\mathfrak{p})$. Hence, $Q_\mathfrak{p}/\mathfrak{p}Q_\mathfrak{p} \simeq M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$. We conclude that if $a \in \ker g$, then $a \in \mathfrak{p}Q_\mathfrak{p} \cap Q$. Since this happens for any prime ideal, we further conclude that $\ker g \subset (\bigcap_{\mathfrak{p} \in \operatorname{Spec} R} \mathfrak{p}Q_\mathfrak{p}) \cap Q$. Recall that Q is a free module so that $Q \simeq R^n$. We get $(\bigcap \mathfrak{p}Q_\mathfrak{p}) \cap Q = (\bigcap \mathfrak{p}R_\mathfrak{p}^n) \cap R^n = ((\bigcap \mathfrak{p}R_\mathfrak{p}) \cap R)^n = (\bigcap \mathfrak{p}R)^n = 0$ since R is reduced. □

We shall find it convenient to "localize" the notion of a kG-module of constant j-rank given in Definition 3.11 as follows.

Definition 4.11. Let G be an infinitesimal group scheme, and let M be a finite dimensional kG-module. For any open subset $U \subset \mathbb{P}(\mathrm{G})$, M is said to be of constant j-rank when restricted to U if $\mathrm{rk}_{k(x)}((\theta_x/s)^j)$ is independent of $x \in U$.

Our next theorem emphasizes the local nature of the concept of constant j-rank.

Theorem 4.12. *Let G be an infinitesimal group scheme, let M be a finite dimensional kG-module, and let $X = \mathbb{P}(G)$. Let $U \subset X$ be a connected open subset, and $\widetilde{\Theta}^j_U : \mathcal{M}_{|U} \to \mathcal{M}(jp^{r-1})_{|U}$ be the restriction to U of the j^{th} iterate of $\widetilde{\Theta}_G$ on $\mathcal{M} = \mathcal{O}_X \otimes M$ as given in (4.5.1). Then the following are equivalent for some fixed j, $1 \leq j < p$:*

1. $\mathrm{Im}\{\widetilde{\Theta}^j_U, \mathcal{M}_{|U}\}$ *is a locally free, coherent $\mathcal{O}_U$-module.*
2. $k(x) \otimes_{\mathcal{O}_X} \mathrm{Im}\{\widetilde{\Theta}^j_G, \mathcal{M}\}$ *has dimension independent of $x \in U$.*
3. $\mathrm{Im}\{\theta^j_x, M_{k(x)}\} \;\simeq\; k(x) \otimes_{\mathcal{O}_X} \mathrm{Im}\{\widetilde{\Theta}^j_G, \mathcal{M}\},\ \forall\ x \in U$
4. *M has constant j-rank when restricted to U.*

Moreover, each of these conditions implies that

5. $\mathrm{Ker}\{\widetilde{\Theta}^j_U, \mathcal{M}_U\}$ *is a locally free, coherent $\mathcal{O}_U$-module.*
6. $\mathrm{Ker}\{\theta^j_x, M_{k(x)}\} \;\simeq\; k(x) \otimes_{\mathcal{O}_X} \mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{M}\},\ \forall\ x \in U.$

Proof. Clearly, (1) implies (2), whereas Proposition 4.10 implies that (2) implies (1).

If we assume (1), we obtain a locally split short exact sequence of coherent $\mathcal{O}_U$-modules

$$0 \;\to\; \mathrm{Ker}\{\widetilde{\Theta}^j_U, \mathcal{M}_{|U}\} \;\to\; \mathcal{M}_{|U} \;\to\; \mathrm{Im}\{\widetilde{\Theta}^j_U, \mathcal{M}_{|U}\} \to 0. \tag{4.12.1}$$

In particular, $\mathrm{Ker}\{\widetilde{\Theta}^j_U, \mathcal{M}_{|U}\}$ is a locally free, coherent $\mathcal{O}_U$-module. Locally on U, $\widetilde{\Theta}^j_U$ on $\mathcal{M}_{|U}$ is isomorphic to the projection

$$pr_2 : \mathrm{Ker}\{\widetilde{\Theta}^j_U, \mathcal{M}_{|U}\} \oplus \mathrm{Im}\{\widetilde{\Theta}^j_U, , \mathcal{M}_{|U}\} \;\to\; \mathrm{Im}\{\widetilde{\Theta}^j_U, , \mathcal{M}_{|U}\}.$$

Since θ^j_x is the base change via $\mathcal{O}_U \to k(x)$ of $\widetilde{\Theta}^j_U$, θ^j_x can be identified with the base change of this projection and thus we may conclude (3).

Let us now assume (3). A simple argument using Nakayama's Lemma as in the proof of Proposition 4.10 implies that the function $x \mapsto \mathrm{Im}\{\theta^j_x, M_{k(x)}\}$ is lower semi-continuous on U whereas the function $x \mapsto k(x) \otimes_{\mathcal{O}_X} \mathrm{Im}\{\widetilde{\Theta}^j_G, \mathcal{M}\}$ is upper semi-continuous on U. Thus, we conclude that each of these functions is constant (since U is connected), thereby concluding (2).

Since $\mathrm{rk}\{(\theta_x/s)^j\} \;=\; \dim_{k(x)}(\mathrm{Im}\{\theta^j_x, M_{k(x)}\})$, (2) and (3) imply (4).

To prove that (4) implies (5), observe that if $f : V \to V$ is an endomorphism of a finite dimensional vector space then $\dim\{\mathrm{Coker}\, f\} = \dim\{\mathrm{Ker}\, f\}$. Define $\mathrm{Coker}\{\theta^j_x, M_{k(x)}\}$ to be $\mathrm{Coker}\{(\theta_x/s)^j : M_{k(x)} \to M_{k(x)}\}$. The assumption that the kG-module M has constant rank (i.e., (4)) implies that

$$\dim_{k(x)}(\mathrm{Coker}\{\theta^j_x, M_{k(x)}\}) \;=\; \dim_{k(x)}(\mathrm{Ker}\{\theta^j_x, M_{k(x)}\})$$

is independent of $x \in U$. On the other hand, the right exactness of $k(x) \otimes_{\mathcal{O}_X} (-)$ applied to

$$\mathcal{M} \;\stackrel{\widetilde{\Theta}^j_G}{\to}\; \mathcal{M}(jp^{r-1}) \to \mathrm{Coker}\{\widetilde{\Theta}^j_G, \mathcal{M}(jp^{r-1})\} \to 0$$

implies that

$$\mathrm{Coker}\{\theta_x^j, M_{k(x)}\} \;=\; k(x) \otimes_{\mathcal{O}_X} \mathrm{Coker}\{\widetilde{\Theta}_G^j, \mathcal{M}(jp^{r-1})\}.$$

Hence, Proposition 4.10 implies that $\mathrm{Coker}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\}$ is a locally free coherent $\mathcal{O}_U$-module whenever M is of constant rank on U. Thus, assuming (4), we obtain a locally split short exact sequence of coherent $\mathcal{O}_U$-modules

$$0 \to \mathrm{Im}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\} \to \mathcal{M}(jp^{r-1})_{|U} \to \mathrm{Coker}\{\widetilde{\Theta}_U^j, \mathcal{M}(jp^{r-1})_{|U}\} \to 0,$$

so that $\mathrm{Im}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\}$ is a locally free, coherent $\mathcal{O}_U$-module. Now, using the the short exact sequence of coherent $\mathcal{O}_U$-modules

$$0 \to \mathrm{Ker}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\} \;\to\; \mathcal{M}_{|U} \;\to\; \mathrm{Im}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\} \to 0,$$

we conclude that (4) implies (5) (i.e., that $\mathrm{Ker}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\}$ is locally free.

Since the short exact sequence (4.12.1) is locally split, applying $k(x) \otimes_{\mathcal{O}_X} -$ to (4.12.1) for any $x \in U$ yields a short exact sequence, thereby implying (6). $\square$

5. Vector bundles for modules of constant j-rank

In this section, we initiate the study of algebraic vector bundles associated to kG-modules of constant j-rank as defined in 3.11. Our constructions have two immediate consequences. The first is that certain kG-modules with the same "local Jordan type" have non-isomorphic associated vector bundles, so that the isomorphism classes of these vector bundles serve as a new invariant. The second is that our construction yields vector bundles on the highly non-trivial projective schemes $\mathbb{P}(\mathrm{G})$.

The reader will find formulas for the ranks of bundles considered, criteria for non-triviality of bundles, a criterion for producing line bundes, a relationship to duality, and another test for the projectivity of kG-modules. We also investigate the dimension of global sections of various bundles.

As in §4, we assume that $\dim V(G) \geq 1$ throughout this section.

The special case in which $U = \mathbb{P}(\mathrm{G})$ of Theorem 4.12 is the following assertion that kG-modules of constant j-rank determine algebraic vector bundles over $\mathbb{P}(\mathrm{G})$.

Theorem 5.1. *Let G be an infinitesimal group scheme, let M be a finite dimensional kG-module, and let $\mathcal{M} = \mathcal{O}_{\mathbb{P}(\mathrm{G})} \otimes M$ be a free coherent sheaf on $\mathbb{P}(\mathrm{G})$. Then M has constant j-rank if and only if $\mathrm{Im}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$ is an algebraic vector bundle on $\mathbb{P}(\mathrm{G})$.*

Consequently, if M has constant j-rank, then $\mathrm{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}$ also is an algebraic vector bundle on $\mathbb{P}(\mathrm{G})$.

Remark 5.2. Unless M is trivial as a kG-module, $\mathrm{Ker}\{\Theta_G : k[V(G)] \otimes M \;\to\; k[V(G)] \otimes M\}$ is not projective as a $k[V(G)]$-module, since the local p-nilpotent operator θ_0 at $0 \in V(G)$ is the 0-map.

We observe the following elementary functoriality of this construction.

Proposition 5.3. *Let $i : H \to G$ be an embedding of infinitesimal group schemes, let M be a finite dimensional kG-module, and let N be the restriction of M to kH. Let $\mathcal{M} = \mathcal{O}_{\mathbb{P}(G)} \otimes M$, and $\mathcal{N} = \mathcal{O}_{\mathbb{P}(H)} \otimes N$. Then for any $j, 1 \leq j < p$, there* *are natural isomorphisms of coherent sheaves on $\mathbb{P}(H)$, where $f : \mathbb{P}(H) \to \mathbb{P}(G)$ is induced by i:*

$$f^* \operatorname{Im}\{\widetilde{\Theta}^j_G, \mathcal{M}\} \;\simeq\; \operatorname{Im}\{\widetilde{\Theta}^j_H, \mathcal{N}\}$$
$$f^* \operatorname{Ker}\{\widetilde{\Theta}^j_G, \mathcal{M}\} \;\simeq\; \operatorname{Ker}\{\widetilde{\Theta}^j_H, \mathcal{N}\}.$$

Proof. The statement follows immediately from the commutativity of the diagram

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}(G)} \otimes M \arrow[r, "\widetilde{\Theta}^j_G"] \arrow[d, "f^*"'] & \mathcal{O}_{\mathbb{P}(G)}(jp^{r-1}) \otimes M \arrow[d, "f^*"'] \\ \mathcal{O}_{\mathbb{P}(H)} \otimes N \arrow[r, "\widetilde{\Theta}^j_H"] & \mathcal{O}_{\mathbb{P}(H)}(jp^{r-1}) \otimes N. \end{tikzcd}$$

The diagram is commutative by Proposition 2.7.3.

□

Example 5.4. For each of our four examples of infinitesimal group schemes (initially investigated in Example 1.4), we give examples of kG-modules of constant Jordan type taken from [6].

(1) Let $\mathfrak{g}$ be a finite dimensional p-restricted Lie algebra of dimension at least 2. For any Tate cohomology class of negative dimension, $\zeta \in \widehat{H}^n(\mathfrak{u}(\mathfrak{g}), k) \simeq \operatorname{Ext}^1_{\mathfrak{u}(\mathfrak{g})}(\Omega^{n-1}(k), k)$, we consider the extension of $\mathfrak{u}(\mathfrak{g})$-modules

$$0 \longrightarrow k \longrightarrow M \longrightarrow \Omega^{n-1}(k) \longrightarrow 0$$

determined by ζ. By [6, 6.3], M is a $\mathfrak{u}(\mathfrak{g})$-module of constant Jordan type. We verify by inspection that the Jordan type of M is $(a, 0, \ldots, 0, 2)$ for some $a > 0$ if n is odd, and $(b, 1, 0, \ldots, 0, 1)$ for some $b > 0$ if n is even (see (3.1.2) for notation).

(2) Let $G = \mathbb{G}_{a(r)}$, and set I equal to the augmentation ideal of $kG \simeq k[u_0, \ldots, u_{p-1}]/(u_0^p, \ldots, u_{p-1}^p)$. As observed in [6], I^i/I^t is a module of constant Jordan type for any $t > i$. According to an unpublished, non-trivial computation of A. Suslin, the only ideals of $k\mathbb{G}_{a(2)}$ which are of constant Jordan type are of the form I^i.

(3) As observed in [6], the n^{th} syzygy module $\Omega^n(k)$, $n \in \mathbb{Z}$, is a module of constant Jordan type for any infinitesimal group scheme G. For n even, $\Omega^n(k)$ has constant Jordan type $(a, 0, \ldots, 0, 1)$ for some $a > 0$; whereas for n odd, $\Omega^n(k)$ has constant Jordan type $(b, p-1, 0, \ldots, 0)$ for some $b > 0$.

(4) For $G = \mathrm{SL}_{2(2)}$, we recall that the cohomology algebra $\mathrm{H}^\bullet(G, k)$ is generated modulo nilpotents by classes $\zeta_1, \zeta_2, \zeta_3 \in \mathrm{H}^2(G, k)$ and classes $\xi_1, \xi_2, \xi_3 \in \mathrm{H}^{2p}(G, k)$ ([16]). As in [6, 6.8], the kG-module

$$M \;\equiv\; \operatorname{Ker}\{\sum \zeta_i + \sum \xi_j : (\Omega^2(k))^{\oplus 3} \oplus (\Omega^{2p}(k))^{\oplus 3} \;\to\; k\}$$

is a kG-module of constant Jordan type $(a, 0, \ldots, 0, 1)$ for some $a > 0$.

We elaborate on the Example 5.4(2), constructing $\mathbb{G}_{a(r)}$-modules of constant j-rank for but not of constant Jordan type.

Example 5.5. We start with the following simple observation. Let $M_1 \subset M_2 \subset M$ be a chain of k-vector spaces, and let ϕ be an endomorphism of M_3 such that $\phi(M_1) \subset M_1$ and $\phi(M_2) \subset M_2$. If $\dim(\operatorname{Ker} \phi_{|_{M_1}}) = \dim(\operatorname{Ker} \phi)$, then $\dim(\operatorname{Ker} \phi_{|_{M_1}}) = \dim(\operatorname{Ker} \phi_{|_{M_2}}) = \dim(\operatorname{Ker} \phi)$.

Let $G = \mathbb{G}_{a(r)}$, and set I equal to the augmentation ideal of $kG \simeq k[u_0, \ldots, u_{p-1}]/(u_0^p, \ldots, u_{p-1}^p)$. Consider any ideal J of kG with the property that $I^i \subset J$ for some i, $i \leq p-1$. Note that for any $\underline{a} \in \mathbb{A}^r$, and any $j \leq p - i$,

$$\dim(\operatorname{Ker}\{\theta_{\underline{a}}^j : I^i \to I^i\}) = pj = \dim(\operatorname{Ker}\{\theta_{\underline{a}}^j : kG \to kG\}). \tag{5.5.1}$$

Indeed, since I^i is a module of constant Jordan type, it suffices to check the statement for $\theta_{\underline{a}} = u_0$ for which it is straightforward. The observation in the previous paragraph together with (5.5.1) and the inclusions $I^i \subset J \subset kG$ imply

$$\dim(\operatorname{Ker}\{\theta_{\underline{a}}^j : J \to J\}) = pj$$

for any $j \leq p - i$ and any $\underline{a} \in \mathbb{A}^r$. Hence, J has constant j-rank for $1 \leq j \leq p - i$.

In the following example, we offer a method applicable to almost all infinitesimal group schemes G of constructing kG-modules which are of constant rank but not constant Jordan type.

Example 5.6. Let G be an infinitesimal group scheme with the property that $V(G)$ has dimension at least 2. Assume that p is odd, and let $n > 0$ be an odd positive integer. Let $\zeta \in \mathrm{H}^n(G, k)$ be a non-zero cohomology class and let M denote the kernel of $\zeta : \Omega^n(k) \to k$. Then M has constant rank but not constant Jordan type. Namely, the local Jordan type of M at $0 \neq v \in V(G)$ is $(a, 0, 1, 0, \ldots, 0)$ if $\zeta(v) \neq 0$, and is $(a-1, 2, 0, \ldots, 0)$ if $\zeta(v) = 0$. These Jordan types have the same rank.

For $G = \mathrm{SL}_{2(1)}$ any module is a module of constant Jordan type (see [6]). We calculate explicitly which bundles correspond the irreducible $\mathrm{SL}_{2(1)}$-modules.

Example 5.7. Let $G = \mathrm{SL}_{2(1)}$, $A = k[V(G)]$, and let M be the canonical rational 2-dimensional SL_2-module. We have $A \simeq k[x, y, z]/(xy + z^2)$. Hence, $\mathbb{P}(\mathrm{G})$ is a smooth projective conic and therefore is isomorphic to $\mathbb{P}^1$. The universal p-nilpotent operator Θ_G is given by the formula $\Theta_G = xe + yf + zh$, where $\langle e, f, h\rangle$ is the standard basis of sl_2 (see Example 2.5(1)). Hence, by Example 2.6, the action of Θ_G on $A \otimes M \simeq A^2$ is given by the matrix

$$\Theta_G : A^2 \xrightarrow{\begin{pmatrix} z & x \\ y & -z \end{pmatrix}} A^2 \, .$$

Let $U_x \subset V(G)$ be the open affine defined by $x \neq 0$. We have

$$\begin{pmatrix} z & x \\ y & -z \end{pmatrix} \sim \begin{pmatrix} z \cdot \frac{z}{x} & x \cdot \frac{z}{x} \\ y & -z \end{pmatrix} \sim \begin{pmatrix} -y & z \\ y & -z \end{pmatrix} \sim \begin{pmatrix} -y & z \\ 0 & 0 \end{pmatrix}$$

in the localization $A[\frac{1}{x}]$. Hence, the rank of Θ_G equals 1 in $A[\frac{1}{x}]$ and the kernel is a free module of rank 1. A similar calculation shows that the same is true on the affine open U_y defined by $y \neq 0$. Since $\mathbb{P}(\mathrm{G}) = U_x \cup U_y$, we conclude that $\operatorname{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}$, where $\mathcal{M} = \mathcal{O}_{\mathbb{P}(\mathrm{G})} \otimes M$, is a locally free sheaf on $\mathbb{P}(\mathrm{G})$ of rank 1. In fact, $\operatorname{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$.

More generally, let S_λ be the irreducible sl_2-module of highest weight λ, $0 \leq \lambda \leq p-1$. The case of the canonical representation considered above corresponds to $\lambda = 1$. Let $v_0, v_1, \dots, v_\lambda$ be a basis for S_λ such that the generators e, f and h of sl_2 act as follows: $hv_i = (\lambda - 2i)v_i$, $ev_i = (\lambda - i + 1)v_{i-1}$ for $i > 0$, $ev_0 = 0$, and $fv_i = (i+1)v_{i+1}$ (see [18, 7.2]). We conclude that the operator

$$\Theta_G : A \otimes S_\lambda \simeq A^{\lambda+1} \longrightarrow A \otimes S_\lambda \simeq A^{\lambda+1}$$

is represented by the matrix

$$\begin{pmatrix} \lambda z & \lambda x & 0 & \dots & \\ y & (\lambda-2)z & (\lambda-1)x & \dots & \\ 0 & 2y & (\lambda-4)z & (\lambda-2)x & \dots \\ \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \lambda y & -\lambda z \end{pmatrix}. \tag{5.7.1}$$

A calculation similar to the special case of $M = S_1$ yields that the rank of this matrix on U_x and U_y is λ. Hence, $\mathrm{Ker}\{\widetilde{\Theta}_G, \mathcal{S}_\lambda\}$, where $\mathcal{S}_\lambda = \mathcal{O}_{\mathbb{P}(\mathrm{G})} \otimes S_\lambda$, is a locally free sheaf of rank 1. Moreover, $\mathrm{Ker}\{\widetilde{\Theta}_G, \mathcal{S}_\lambda\} \simeq \mathcal{O}_{\mathbb{P}^1}(-\lambda)$.

One may readily determine the rank of various bundles of $\mathbb{P}(G)$ associated to modules of constant Jordan type using the next proposition.

Proposition 5.8. *Let G be an infinitesimal group scheme, let M a kG-module of constant Jordan type $\sum_{i=1}^p a_i[i]$, and let $\mathcal{M}$ denote the free $\mathcal{O}_{\mathbb{P}(\mathrm{G})}$-module $\mathcal{O}_{\mathbb{P}(\mathrm{G})} \otimes M$. Then for any $j, 1 \leq j < p$,*

$$\mathrm{rk}(\mathrm{Im}\{\widetilde{\Theta}^j_G, \mathcal{M}\}) \;=\; \sum_{i=j+1}^{p} a_i(i-j). \tag{5.8.1}$$

In particular,

$$\mathrm{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\} \subset \mathrm{Ker}\{\widetilde{\Theta}^2_G, \mathcal{M}\} \subset \cdots \subset \mathrm{Ker}\{\widetilde{\Theta}^{p-1}_G, \mathcal{M}\} \subset \mathcal{M}$$

is a chain of $\mathcal{O}_{\mathbb{P}(\mathrm{G})}$-submodules with $\mathrm{rk}(\mathrm{Ker}\{\widetilde{\Theta}^{j-1}_G, \mathcal{M}\}) < \mathrm{rk}(\mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{M}\})$ if and only if $a_i \neq 0$ for some $1 \leq j \leq i \leq p$.

Proof. The formula (5.8.1) is the formula for the rank of u^j on the $k[u]/u^p$-module $\oplus_i (k[u]/u^i)^{\oplus a_i}$ of Jordan type $\sum_{i=1}^p a_i[i]$. This is therefore the dimension of the image of θ_v, $0 \neq v \in V(G)$ on $M_{k(v)}$, and thus the rank of the vector bundle $\mathrm{Im}\{\widetilde{\Theta}^j_G, \mathcal{M}\}$. □

The following class of modules, of interest in its own right, is currently being studied by Jon Carlson and the authors.

Definition 5.9. Let G be an infinitesimal group scheme, M a finite dimensional kG-module, and $j < p$ a positive integer. We say that a kG-module M has the *constant j-image property* if there exists a subspace $I(j) \subset M$ such that for every $v \neq 0$ in $V(G)$, the image of $\theta^j_v : M_{k(v)} \to M_{k(v)}$ equals $I(j)_{k(v)}$. Similarly, we say that M has *constant j-kernel property* if there exists some submodule $K(j) \subset M$ such that for every $v \neq 0$ in $V(G)$, the kernel of $\theta^j_v : M_{k(v)} \to M_{k(v)}$ equals $K(j)_{k(v)}$.

We see that these modules are precisely those whose associated vector bundles are trivial vector bundles.

Proposition 5.10. *Let G be an infinitesimal group scheme, let M be a finite dimensional kG-module which is of constant j-rank. Then the algebraic vector bundle $\mathrm{Im}\{\widetilde{\Theta}^j_G, \mathcal{M}\}$ is trivial (i.e., a free coherent sheave) on $\mathbb{P}(\mathrm{G})$ if and only if M has the constant j-image property. Similarly, $\mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{M}\}$ is trivial if and only if M has the constant j-kernel property.*

Proof. If M has a constant j-image property then $\mathrm{Im}\{\widetilde{\Theta}^j_G, \mathcal{M}\}$ is a free $\mathcal{O}_X$-module generated by $I(j)$. Conversely, assume that $\mathrm{Im}\{\widetilde{\Theta}^j_G, \mathcal{M}\}$ is a free $\mathcal{O}_X$-module. Then there exists a subspace $I(j) \subset M = \Gamma(X, \mathcal{M})$ which maps to and spans each fiber $\mathrm{Im}\{\theta^j_v, M_{k(v)}\}$, for $0 \neq v \in V(G)$. The argument for kernels is similar. $\square$

Remark 5.11. We point out the properties of constant j-image and constant j-kernel are independent of each other. Consider the module $M^\#$ of Example 6.1. As shown in that example, $\mathrm{Ker}\{\widetilde{\Theta}_G, \mathcal{M}^\#\}$ is locally free of rank 2 but not free, since the global sections have dimension one. On the other hand, $\mathrm{Im}\{\widetilde{\Theta}_G, \mathcal{M}^\#\}$ is a free $\mathcal{O}_X$-module generated by the global section n_3. In particular, $M^\#$ has constant image property but not constant kernel property.

For the module M of Example 6.1, the sheaf $\mathrm{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}$ is free of rank 2 whereas $\mathrm{Im}\{\widetilde{\Theta}_G, \mathcal{M}\}$ is locally free of rank 1 but not free since it does not have any global sections. Hence, M has a constant kernel property but not constant image property.

We consider an analogue of the sheaf construction of Duflo-Serganova for Lie superalgebras [8]. This construction enables us to produce additional algebraic vector bundles on $\mathbb{P}(\mathrm{G})$. We implicitly use the observation $\widetilde{\Theta}^p_G = 0$.

Definition 5.12. Let G be an infinitesimal group scheme, let X denote $\mathbb{P}(\mathrm{G})$, and consider a finite dimensional kG-module M. Denote $\mathcal{O}_X \otimes M$ by $\mathcal{M}$. For any i, $1 \leq i \leq p-1$, we consider the following coherent $\mathcal{O}_X$-modules, subquotients of $\mathcal{M}$:

$$\mathcal{M}^{[i]} \;\equiv\; \mathrm{Ker}\,\widetilde{\Theta}^i_G/\,\mathrm{Im}\,\widetilde{\Theta}^{p-i}_G \;\equiv\; \mathrm{Ker}\{\widetilde{\Theta}^i_G, \mathcal{M}\}/\,\mathrm{Im}\{\widetilde{\Theta}^{p-i}_G, \mathcal{M}(-(p-i)p^{r-1})\}.$$

The following simple lemma helps to motivate these subquotients.

Lemma 5.13. *Let V be a finite dimensional $k[t]/t^p$-module, and let $JType(V,t) = (a_p, \dots, a_1)$ (using the notation introduced in (3.1.2)). Let*

$$V^{[j]} = \mathrm{Ker}\{t^j : V \to V\}/\,\mathrm{Im}\{t^{p-j} : V \to V\}$$

for $j \leq p-1$. Then

$$\dim(V^{[j]}) \;=\; \sum_{1 \leq i \leq j} i a_i + \sum_{i > j} j a_i - \sum_{i+j>p} (i+j-p) a_i.$$

In particular, V is projective as a $k[t]/t^p$-module if and only if $V^{[1]} \;=\; 0$.

Furthermore, for $j \leq p-1$, $V^{[j]} \simeq V^{[p-j]}$ as $k[t]/t^p$-modules.

As seen in the next proposition, these subquotients can provide additional examples of algebraic vector bundles over $\mathbb{P}(\mathrm{G})$.

Proposition 5.14. *Let G be an infinitesimal group scheme and let M be a finite dimensional kG-module which is of constant j-rank and constant $(p-j)$-rank for some j, $1 \leq j < p$. Then $\mathcal{M}^{[j]}$ is a locally free $\mathcal{O}_X$-module and $k(x) \otimes_{\mathcal{O}_X} \mathcal{M}^{[j]} \to M^{[j]}_{k(x)}$ is an isomorphism for all $x \in X \equiv \mathbb{P}(\mathrm{G})$.*

Proof. For any $x \in X$, consider the map of exact sequences

$$\begin{tikzcd}
k(x) \otimes_{\mathcal{O}_X} \operatorname{Im} \theta_x^{p-j} \arrow[r] \arrow[d] & k(x) \otimes_{\mathcal{O}_X} \operatorname{Ker} \theta_x^{j} \arrow[r] \arrow[d] & k(x) \otimes_{\mathcal{O}_X} \mathcal{M}^{[j]} \arrow[r] \arrow[d] & 0 \\
\operatorname{Im} \theta_x^{p-j} \arrow[r] & \operatorname{Ker} \theta_x^{j} \arrow[r] & M_{k(x)}^{[j]} \arrow[r] & 0.
\end{tikzcd}$$

where $\operatorname{Im} \theta_x$ (respectively, $\operatorname{Ker} \theta_x$) is an abbreviation for $\operatorname{Im}\{\theta_x, M_{k(x)}\}$ (respectively, $\operatorname{Ker}\{\theta_x, M_{k(x)}\}$). The left and middle vertical maps are isomorphisms by Theorem 4.12. Thus, the 5-Lemma implies that the right vertical arrow is also an isomorphism. □

We give an application of this $(-)^{[1]}$ construction to endotrivial modules. An interested reader can compare our construction to [1]. Recall that a module M of a finite group scheme G is endotrivial if $\operatorname{End}_k(M) \simeq k + \operatorname{proj}$. It was shown in [6, §5] that an endotrivial module is a module of constant Jordan type with possible types $[1] + \operatorname{proj}$ and $[p-1] + \operatorname{proj}$.

Proposition 5.15. *Let G be an infinitesimal group scheme, let M be a finite dimensional modle of constant Jordan type, and set $\mathcal{M} = \mathcal{O}_{\mathbb{P}(G)} \otimes M$. Then $\mathcal{M}^{[1]}$ is a line bundle (i.e., an algebraic vector bundle of rank one) if and only if M is endotrivial.*

Proof. The sheaf $\mathcal{M}^{[1]}$ is locally free by Proposition 5.14. Let $\sum_{i=1}^{p} a_i[i]$ be the Jordan type of M. Proposition 5.14 implies that the rank of the vector bundle $\mathcal{M}^{[1]}$ equals $\sum_{i=0}^{p-1} a_i$. Hence, $\mathcal{M}^{[1]}$ is a line bundle if and only if the Jordan type of M has only one non-projective block. A theorem of D. Benson [3] states that modules of constant Jordan type with unique non-projective block must be of type $[1] + \operatorname{proj}$ or $[p-1] + \operatorname{proj}$. By [6, §5], this happens if and only if M is endotrivial. □

We next give a global version of the observation in Lemma 5.13 that $V^{[j]} \simeq V^{[p-j]}$ for $j \leq p-1$.

Proposition 5.16. *Let G be an infinitesimal group scheme. Let M be a finite dimensional kG-module which is of constant j-rank and of constant $(p-j)$-rank for some j, $1 \leq j < p$, and let $N = Hom_k(M, k)$ denote the k-linear dual of M. Then the locally free, coherent $\mathcal{O}_X$-module $\mathcal{N}^{[p-j]}$ is naturally isomorphic to the $\mathcal{O}_X$-linear dual $Hom_{\mathcal{O}_X}(\mathcal{M}^{[j]}, \mathcal{O}_X)$ of $\mathcal{M}^{[j]}$, where $X \equiv \mathbb{P}(\mathrm{G})$. Here, $\mathcal{M} = \mathcal{O}_X \otimes M$, $\mathcal{N} = \mathcal{O}_X \otimes N$.*

Proof. The $\mathcal{O}_X$-linear dual of the complex of $\mathcal{O}_X$-modules

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-(p-j)p^{r-1}) \xrightarrow{\widetilde{\Theta}_G^{p-j}} \mathcal{M} \xrightarrow{\widetilde{\Theta}_G^{j}} \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(jp^{r-1})$$

is the complex

$$\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{O}_X((p-j)p^{r-1}) \xleftarrow{\widetilde{\Theta}_G^{p-j}} \mathcal{N} \xleftarrow{\widetilde{\Theta}_G^{j}} \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-jp^{r-1}) \, .$$

A similar statement applies with θ_v in place of $\widetilde{\Theta}_G$.

For any scheme Y and any complex of $\mathcal{O}_Y$-modules

$$S_1 \xrightarrow{f} S_2 \xrightarrow{g} S_3$$

with $\mathcal{O}_Y$-linear dual

$$Hom_{\mathcal{O}_Y}(S_1, \mathcal{O}_Y) \xleftarrow{f^\#} Hom_{\mathcal{O}_Y}(S_2, \mathcal{O}_Y) \xleftarrow{g^\#} Hom_{\mathcal{O}_Y}(S_3, \mathcal{O}_Y)\ ,$$

there is a natural (evident) pairing

$$(\mathrm{Ker}\{g\}/\,\mathrm{Im}\{f\}) \otimes (\mathrm{Ker}\{f^\#\}/\,\mathrm{Im}\{g^\#\}) \longrightarrow \mathcal{O}_Y. \tag{5.16.1}$$

One readily verifies that (5.16.1) is a perfect pairing if $\mathcal{O}_Y$ is a field, so that (5.16.1) induces an isomorphism for any $x \in X$ between

$$M_{k(x)}^{[j]} \;=\; \mathrm{Ker}\,\theta_x^j/\,\mathrm{Im}\,\theta_x^{p-j}$$

and the $k(x)$-linear dual of

$$N_{k(x)}^{[p-j]} \;=\; \mathrm{Ker}\,\theta_x^{p-j}/\,\mathrm{Im}\,\theta_x^j.$$

On the other hand, Theorem 4.12 and the right exactness of $k(x) \otimes_{\mathcal{O}_X} (-)$ imply that

$$M_{k(x)}^{[j]} \;\simeq\; k(x) \otimes_{\mathcal{O}_X} \mathcal{M}^{[j]}$$

for every $x \in X$ and similarly for $N_{k(x)}^{[p-j]}$. Thus, the map induced by (5.16.1) (with $X = Y$) is an isomorphism by Proposition 4.10. □

Consideration of $\mathcal{M}^{[1]}$ leads to another characterization of projective kG-modules.

Proposition 5.17. *Let G be an infinitesimal group scheme and let M be a finite dimensional kG-module. Then M is projective if and only if M has constant rank, has constant $(p-1)$–rank, and satisfies $\mathcal{M}^{[1]} \;=\; 0$ (where $\mathcal{M}^{[1]}$ is defined in 5.12).*

Proof. Assume that M is a projective kG-module. Then M has constant Jordan type (which is some multiple of $[p]$), and hence has constant rank and constant $(p-1)$-rank. For any $x \in \mathbb{P}(\mathrm{G}) = X$, $\theta_x^*(M_{k(x)})$ is a free $k(x)[t]/t^p$-module of rank equal to $\frac{\dim(M)}{p}$. If we lift a basis of this free module to $\mathcal{M}_{(x)} \equiv \mathcal{O}_{X,x} \otimes_{\mathcal{O}_X} \mathcal{M}$, then an application of Nakayama's Lemma tells us that $\mathcal{M}_{(x)}$ is free as an $\mathcal{O}_{X,x}[t]/t^p$-module. This readily implies that $(\mathcal{M}_{(x)})^{[p-1]} \equiv \mathrm{Ker}\{\widetilde{\Theta}_{G,(x)}^{p-1}, \mathcal{M}_{(x)}\}/\,\mathrm{Im}\{\widetilde{\Theta}_{G,(x)}, \mathcal{M}_{(x)}\}$ vanishes. Using the exactness of localization, we conclude that $(\mathcal{M}^{[p-1]})_{(x)} = (\mathcal{M}_{(x)})^{[p-1]}$. Consequently, $\mathcal{M}^{[p-1]} = 0$. By Proposition 5.16, we conclude that $\mathcal{M}^{[1]} = 0$.

Conversely, if M has constant rank and constant $(p-1)$-rank and if $\mathcal{M}^{[1]} = 0$, then Proposition 5.14 tells us that $M_{k(x)}^{[1]} \equiv \mathrm{Ker}\,\theta_x/\,\mathrm{Im}\,\theta_x^{p-1}$ equals 0 for all $x \in X$. Lemma 5.13 thus implies that each $M_{k(x)}$ is projective, so that the local criterion for projectivity [22] implies that M is projective. □

One very simple invariant of the algebraic vector bundle $\mathrm{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$ is the dimension of its vector space of global sections. The following proposition gives a method of determining global sections.

Proposition 5.18. *Let G be an infinitesimal group scheme, and assume that $V(G)$ is reduced. Let M be a finite dimensional kG-module and let $\mathcal{M} = \mathcal{O}_{\mathbb{P}(\mathrm{G})} \otimes M$. Then*

$$\Gamma(\mathbb{P}(\mathrm{G}), \mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{M}\}) \ \subset \ M$$

consists of those $m \in M$ such that $\theta^j_x(m) = 0$ for all $x \in \mathbb{P}(\mathrm{G})$.

In particular, if KG is generated by $\theta_v \in k(v)G,\ v \in V(G)$, where K the field of fractions of $k[V(G)]$, then

$$\Gamma(\mathbb{P}(\mathrm{G}), \mathrm{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}) \ = \ \mathrm{H}^0(G, M).$$

Proof. Recall that $\mathbb{P}(\mathrm{G})$ is connected by [6, 3.4] and thus $\Gamma(\mathbb{P}(\mathrm{G}), \mathcal{M}) = M$. Under this identification, the global sections of $\mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{M}\}$ coincide with the subset

$$\{m \in M \,|\, \Theta^j_G(1 \otimes m) = 0\}.$$

where $\Theta_G : k[V(G)] \otimes M \longrightarrow k[V(G)] \otimes M$ is the universal p-nilpotent operator acting on $k[V(G)] \otimes M$ as defined in(2.3.2). Since $V(G)$ is reduced, we have $\Theta^j_G(1 \otimes m) = 0$ if and only if $\theta^j_v(1 \otimes m) = \Theta^j_G(1 \otimes m) \otimes_{k[V(G)]} k(v) = 0$ for any $v \in V(G)$. Hence, $m \in \Gamma(\mathbb{P}(\mathrm{G}), \mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{M}\})$ if and only if $m \in \mathrm{Ker}\{\theta^j_v, M_{k(v)}\}$ for any $v \in V(G)$.

The second assertion concerning the global sections of $\mathrm{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}$ follows immediately upon taking $j = 1$. □

Combining Propositions 5.8 and 5.18 in the special case $j = 1$ yields the following criterion for the non-triviality of $\mathrm{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}$.

Corollary 5.19. *Let G be an infinitesimal group scheme such that $V(G)$ is reduced and positive dimensional, and assume that KG is generated by $\theta_v \in k(v)G,\ v \in V(G)$ for K the field of fractions of $k[V(G)]$. Let M a finite dimensional kG-module of constant Jordan type $\sum_i a_i[i]$. If*

$$\dim \mathrm{H}^0(G, M) \ < \sum_{i=1}^{p} a_i,$$

that $\mathrm{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}$ is a non-trivial algebraic vector bundle over $\mathbb{P}(\mathrm{G})$

Proof. By Proposition 5.8, the dimension of the fibers of $\mathrm{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}$ is $\dim M - \sum_{i=2}^p a_i(i-1) = \sum_{i=1}^p a_i$. By Proposition 5.18, the global sections of $\mathrm{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}$ equal $\mathrm{H}^0(G, M)$. Hence, the inequality $\dim \mathrm{H}^0(G, M) \ < \sum_{i=1}^p a_i$ implies that the dimension of the global sections is less than the dimension of the fibers. Therefore, the sheaf is not free.

□

The following two lemmas will be applied to prove Proposition 5.22

Lemma 5.20. *Let R be a local noetherian ring with residue field k and let M be a finite $R[t]/t^p$-module which is free as an R-module. If $k \otimes_R M$ is a free $k[t]/t^p$-module, then M is free as an $R[t]/t^p$-module.*

Proof. Let $m_1, \ldots, m_s \in M$ be such that $\overline{m}_1, \ldots, \overline{m}_s$ forms a basis for $k \otimes_R M$ as a $k[t]/t^p$-module. Let Q be a free $R]t]/t^p$-module of rank s with basis $q_1, \ldots, q_s$ and consider the $R[t]/t^p$-module homomorphism $f : Q \to M$ sending q_i to m_i.

By Nakayama's Lemma, $f : Q \to M$ is surjective. Because M is free as an R-module, applying $k \otimes_R -$ to the short exact sequence $0 \to \mathrm{Ker}\{f\} \to Q \to M \to 0$ determines the short exact sequence

$$0 \to k \otimes_R \mathrm{Ker}\{f\} \ \to \ k \otimes_R Q \ \to \ k \otimes_R M \to 0.$$

Consequently, $k \otimes_R \mathrm{Ker}\{f\} = 0$, so that another application of Nakayama's Lemma implies that $\mathrm{Ker}\{f\} = 0$. Hence, f is an isomorphism, and thus M is free as an $R[t]/t^p$-module. □

Lemma 5.21. *Let G be an infinitesimal group scheme and M be a finite dimensional kG-module. Set $A = k[V(G)]$; for any $f \in A$, set $A_f = A[1/f]$. Assume that $\mathrm{Spec}\, A_f \subset V(G)$ has empty intersection with $V(G)_M$. Then $(\mathcal{U}_G \circ \epsilon)^*(A_f \otimes M)$ is a projective $A_f[t]/t^p$ -module.*

Proof. By definition, $V(G)_M$ consists of those points $v \in V(G)$ such that $\theta_v^*(M_{(k(v)})$ is not free as a $k(v)[t]/t^p$-module. By the universal property of $\mathcal{U}_G \circ \epsilon$, the assumption that $\mathrm{Spec}\, A_f \cap V(G)_M = \emptyset$ implies for every point $v \in \mathrm{Spec}\, A_f$ that $\theta_v^*(M_{k(v)}) = k(v) \otimes_{A_f} (\mathcal{U}_G \circ \epsilon)^*(A_f \otimes M))$ is free as a $k(v)[t]/t^p$-module. Let $A_{(v)}$ denote the localization of A at v. Then Lemma 5.20 implies for every point $v \in \mathrm{Spec}\, A_f$ that the localization $A_{(v)} \otimes_{A_f} (\mathcal{U}_G \circ \epsilon)^*(A_f \otimes M))$ is free as a $A_{(v)}[t]/t^p$-module. This implies that $A_f \otimes M$ is projective (since projectivity of a module over a commutative ring is determined locally). □

We conclude with a property of the (projectivized) rank variety $\mathbb{P}(\mathrm{G})_M$ of a kG-module M.

Proposition 5.22. *Let G be an infinitesimal group scheme, M be a finite dimensional kG-module, and set $\mathcal{M} = \mathcal{O}_{\mathbb{P}(\mathrm{G})} \otimes M$. Then*

$$\mathrm{Supp}_{\mathcal{O}_{\mathbb{P}(\mathrm{G})}}(\mathcal{M}^{[1]}) \ \subset \ \mathbb{P}(\mathrm{G})_M,$$

where $\mathrm{Supp}_{\mathcal{O}_{\mathrm{Proj}(V)}}(\mathcal{M}^{[1]})$ is the support of the coherent sheaf $\mathcal{M}^{[1]}$ (the closed subset of points $x \in \mathbb{P}(\mathrm{G})$ at which $\mathcal{M}^{[1]}_{(x)} \neq 0$).

Proof. Let A denote $k[V(G)]$ and let X denote $\mathbb{P}(\mathrm{G})$. Consider some $x \notin X_M$ and choose some homogeneous polynomial $F \in A$ vanishing on X_M such that $F(x) \neq 0$. Thus, $x \in \mathrm{Spec}(A_F)_0 \subset X$ and $\mathrm{Spec}(A_F)_0 \cap X_M = \emptyset$, where $(A_F)_0$ denote the elements of degree 0 in the localization $A_F = A[1/F]$. It suffices to prove that $x \notin \mathrm{Supp}_{\mathcal{O}_X}(\mathcal{M}^{[1]}) \cap \mathrm{Spec}(A_F)_0$. Equivalently, it suffices to prove that $v \notin \mathrm{Supp}_{A_F}(A_F \otimes M)^{[1]}$ for some $v \in \mathrm{Spec}\, A_F$ mapping to x.

By Lemma 5.21, $(\mathcal{U}_G \circ \epsilon)^*(A_F \otimes M)$ is a projective $A_F[t]/t^p$ -module. This implies that $(\mathcal{U}_G \circ \epsilon)^*(A_F \otimes M)^{[1]} = 0$, and thus that $v \notin \mathrm{Supp}_{A_F}(A_F \otimes M)^{[1]}$. □

Remark 5.23. The reverse inclusion $\mathbb{P}(\mathrm{G})_M \subset \mathrm{Supp}_{\mathcal{O}_{\mathbb{P}(\mathrm{G})}}(\mathcal{M}^{[1]})$ seems closely related to the condition that $k(x) \otimes_{\mathcal{O}_{\mathbb{P}(\mathrm{G})}} \mathrm{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\} \to \mathrm{Ker}\{\theta_x, M_{k(x)}\}$ be surjective.

6. Examples and calculations with bundles

In this final section, we investigate numerous specific examples. The case in which G equals either $\mathbb{G}_{a(1)}^{\times 2}$ or $\underline{sl}_2$ is particularly amenable to computation for $\mathbb{P}(G)$ is isomorphic to $\mathbb{P}^1$. We are intrigued by projective kG-modules, examples of modules of constant Jordan type which are not distinguished by support varieties. We see that our techniques establish lower bounds for the rank of $K_0(\mathbb{P}(G))$, a non-trivial result in view of the fact that $\mathbb{P}(G)$ is typically singular. Specific examples which lead to interesting bundles are "zig-zag modules" and syzygies.

As we see in the following simple example, the isomorphism type of the vector bundles discussed in Theorem 5.1 can be used to distinguish certain kG-modules which have the same local Jordan type. We remind the reader that the local Jordan type of a finite dimensional kG-module M of constant Jordan type is the same as that of its linear dual $M^\#$.

Example 6.1. Let $G = \mathbb{G}_{a(2)}$, so that $k\mathbb{G}_{a(2)} = k[u_0, u_1]/(u_0^p, u_1^p)$, $V(\mathbb{G}_{a(2)}) = \mathbb{A}^2$, $A = k[V(\mathbb{G}_{a(2)})] = k[x_0, x_1]$ graded so that x_0 is given degree 1 and x_1 is given degree p. Then $\Theta_{\mathbb{G}_{a(2)}} = x_1u_0 + x_0^pu_1 \in A[u_0, u_1]/(u_0^p, u_1^p)$, and $\mathbb{P}(\mathrm{G}) = \mathbb{P}^1$. We consider the 3-dimensional kG-module M of constant Jordan type $(0, \ldots, 0, 1, 1)$ and its linear dual $M^\#$, which we represent diagrammatically as follows:

$$\begin{tikzcd} & m_1 \arrow[dl, "y"'] \arrow[dr, "x"] & \\ \mathfrak{m}_2 & & m_3 \end{tikzcd} \qquad \begin{tikzcd} n_1 \arrow[dr, "x"'] & & n_2 \arrow[dl, "y"] \\ & n_3 & \end{tikzcd}$$

Let $\mathcal{M}$ denote the free coherent $\mathcal{O}_{\mathbb{P}^1}$-module $\mathcal{O}_{\mathbb{P}^1} \otimes M$, and let $\mathcal{M}^\# = \mathcal{O}_{\mathbb{P}^1} \otimes M^\#$. To determine $\operatorname{Ker}\{\widetilde{\Theta}_{\mathbb{G}_{a(2)}}, \mathcal{M}\}$ and $\operatorname{Ker}\{\widetilde{\Theta}_{\mathbb{G}_{a(2)}}, \mathcal{M}^\#\}$, we identify the fibers of these kernels. Let $x = [a : b] \in \mathbb{P}^1$, and let (a, b) be a point on $\mathbb{A}^2$ projecting to x. We have $\theta_{(a,b)} = bu_0 + a^pu_1$ and, hence, $\operatorname{Ker}\{\theta_{(a,b)} : M_{k(x)} \to M_{k(x)}\}$ is spanned by m_2, m_3. On the other hand, $\operatorname{Ker}\{\theta_{(a,b)}, M^\#_{k(x)}\}$, is spanned by n_3 and any linear combination $cn_1 + dn_2$ such that $bc + a^pd = 0$.

We conclude that $\operatorname{Ker}\{\widetilde{\Theta}_{\mathbb{G}_{a(2)}}, \mathcal{M}\} \simeq \mathcal{O}^2_{\mathbb{P}^1}$. On the other hand, $\operatorname{Ker}\{\widetilde{\Theta}_{\mathbb{G}_{a(2)}}, \mathcal{M}^\#\}$ satisfies

$$\dim(\Gamma(\mathbb{P}^1, \operatorname{Ker}\{\widetilde{\Theta}_{\mathbb{G}_{a(2)}}, \mathcal{M}^\#\})) \;=\; 1.$$

Thus,

$$\operatorname{Ker}\{\widetilde{\Theta}_{\mathbb{G}_{a(2)}}, \mathcal{M}\} \;\not\simeq\; \operatorname{Ker}\{\widetilde{\Theta}_{\mathbb{G}_{a(2)}}, \mathcal{M}^\#\}.$$

We next give a somewhat more interesting example of pairs of modules of the same constant Jordan type with different associated bundles.

Example 6.2. As seen in [6], any rational SL_2-module M restricts to a $\mathfrak{u}(sl_2)$-module of constant Jordan type. Let S_λ be the irreducible SL_2-module of highest weight $\lambda, 0 < \lambda < p$ (of dimension $\lambda + 1$) and let $M_\lambda = S^p(S_\lambda)$, the p^{th} symmetric power of S_λ. Since S_λ is self-dual, the dual of M_λ is $N_\lambda = \Gamma^p(S_\lambda)$, the p^{th} divided power of S_λ. The modules M_λ, N_λ fit in a short exact sequence of rational SL_2-modules

$$0 \longrightarrow S_\lambda^{(1)} \longrightarrow M_\lambda \longrightarrow N_\lambda \longrightarrow S_\lambda^{(1)} \longrightarrow 0.$$

Here, $S_\lambda^{(1)}$ is the first Frobenius twist of S_λ, thus trivial as a $\mathfrak{u}(sl_2)$-module.

The projectivized null cone $X = \operatorname{Proj} N(sl_2)$ is a rational conic, whose elements can be viewed as homothety classes of non-zero nilpotent elements. Let $\mathcal{M}$, $\mathcal{N}$ denote the free $\mathcal{O}_X$-modules $\mathcal{O}_X \otimes M$, $\mathcal{O}_X \otimes N$. If $0 \neq v \in N(sl_2)$ is viewed as a non-zero nilpotent element in sl_2, then θ_v on $M_{k(v)}$ as given by the action of the corresponding nilpotent element. Thus, for an element in $\Gamma(X,\mathcal{M}) = S^p(S_\lambda)$ to lie in $\Gamma(X, \mathrm{Ker}\{\widetilde{\Theta}_{\underline{s}l_2}, \mathcal{M}\}) \subset \Gamma(X,\mathcal{M})$, it is necessary and sufficient for that element to be sl_2-invariant. We conclude that $\Gamma(X, \mathrm{Ker}\{\widetilde{\Theta}_{\underline{s}l_2}, \mathcal{M}\}) = S_\lambda^{(1)}$.

Similarly, for an element in $\Gamma(X,\mathcal{N}) = \Gamma^p(S_\lambda)$ to lie in $\Gamma(X, \mathrm{Ker}\{\widetilde{\Theta}_{\underline{s}l_2}, \mathcal{N}\})$, it is necessary and sufficient for that element to be sl_2-invariant. Yet the sl_2-invariants of $\Gamma^p(S_\lambda)$ are 0.

Hence, the dual modules M_λ and N_λ have the same local Jordan type but non-isomorphic associated bundles $\mathrm{Ker}\{\widetilde{\Theta}_{\underline{s}l_2}, \mathcal{M}\}$ and $\mathrm{Ker}\{\widetilde{\Theta}_{\underline{s}l_2}, \mathcal{N}\}$.

For any finite-dimensional kG-module M and any j, $1 \leq j \leq p$, we may consider

$$\rho_j(M) \;\equiv\; \dim \Gamma(\mathbb{P}(\mathrm{G}), \mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{M}\}) \tag{6.2.1}$$

where $\mathcal{M} = \mathcal{O}_{\mathbb{P}(\mathrm{G})} \otimes M$. We make this explicit for projective kG-modules, a class of kG-modules for which rank varieties give no information.

Proposition 6.3. *Let G be an infinitesimal group scheme. Then sending a finitely generated projective kG-module P to $(\rho_1(P), \dots, \rho_p(P)) \in \mathbb{N}^p$ determines a covariantly functorial homomorphism*

$$\rho = (\rho_1, \dots, \rho_p) : K_0(kG) \;\to\; \mathbb{Z}^p.$$

Proof. To prove that ρ is well defined on $K_0(G)$, it suffices to observe that each ρ_j is additive and that short exact sequences of projective kG-modules split. □

Proposition 6.4. *Let $G = \underline{s}l_2$ be the infinitesimal group scheme associated to the restricted Lie algebra sl_2. The homomorphism ρ of Proposition 6.3 is a rational isomorphism:*

$$\rho_{\mathbb{Q}} : K_0(\mathfrak{u}(sl_2)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}^p.$$

Proof. Let $X = \mathbb{P}(\underline{s}l_2)$, which is homeomorphic to $\mathbb{P}^1$ as seen in Example 5.7. It suffices to prove the statement of the Proposition once we extend scalars to $\overline{k}$. Hence, we may assume that k is algebraically closed.

Let M be a finite-dimensional rational SL_2-module, and let $\Theta_G : k[V(G)] \otimes M \to k[V(G)] \otimes M$ be the universal p-nilpotent operator. We first show that the kernel of Θ_G restricted to $1 \otimes M$ is an SL_2-stable subspace of M. Let $1 \otimes m \in 1 \otimes M$ be in the kernel of Θ_G, and let $g \in \mathrm{SL}_2(k)$. Let $v \in N(sl_2)\backslash\{0\}$. We have

$$\theta_v(1 \otimes gm) = g(\theta_v^{g^{-1}}(1 \otimes m)) = g\theta_{v^{g^{-1}}}(1 \otimes m)$$

where the actions of $\mathrm{SL}_2(k)$ on $kG \simeq u(sl_2)$ and on $N(sl_2)$ are induced by the adjoint action on sl_2. Iterating the formula, we obtain

$$\theta_v^j(1 \otimes gm) = g\theta^j_{v^{g^{-1}}}(1 \otimes m). \tag{6.4.1}$$

Hence, we have the following equalities:

$$\{m \in M \,|\, \Theta^j_G(1 \otimes m) = 0\} = \bigcap_{0 \neq v \in V(G)} \{m \in M \,|\, \theta^j_v(1 \otimes m) = 0\} =$$

$$\bigcap_{0 \neq v^{g^{-1}} \in V(G)} \{m \in M \,|\, g\theta^j_{v^{g^{-1}}}(1 \otimes m) = 0\} = \bigcap_{0 \neq v \in V(G)} \{m \in M \,|\, \theta^j_v(1 \otimes gm) = 0\} =$$

$$\{m \in M \,|\, \Theta^j_G(1 \otimes gm) = 0\},$$

where the first and the last equality follows from Proposition 5.18, the second equality follows from the transitivity of the adjoin action of $\mathrm{SL}_2(k)$ on $N(sl_2)$, and the third equality is an application of (6.4.1). We conclude that $\{m \in M \,|\, \Theta^j_G(1 \otimes m) = 0\}$ is an $\mathrm{SL}_2(k)$ stable subspace of M. Therefore, $\Gamma(X, \mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{O}_X \otimes M\})$ is a G-stable subspace of M. Here, $X = \mathbb{P}(\underline{s}l_2)$.

The decomposition series of the projective cover P_λ of the irreducible sl_2-module S_λ of highest weight λ, $0 \leq \lambda < p-1$, is represented by the following diagram (see [11, 2.4]):

$$\begin{tikzcd} & S_\lambda \arrow[dl] \arrow[dr] & \\ S_{p-2-\lambda} \arrow[dr] & & S_{p-2-\lambda} \arrow[dl] \\ & S_\lambda & \end{tikzcd}$$

On the other hand, $P_{p-1} = S_{p-1}$ is the Steinberg module of dimension p.

Since $\Gamma(X, \mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{O}_X \otimes P_\lambda\})$ is a G-stable subspace of P_λ, it is non-trivial if and only if it contains the socle S_λ. The simple module S_λ belongs to the kernel of Θ^j_G if and only if it is annihilated by j-th iterations of all nilpotent elements of sl_2, which happens if and only if $j > \lambda$.

We conclude that $\rho_j(P_\lambda) = 0$ for $j \leq \lambda$ and $\rho_{\lambda+1}(P_\lambda) > 0$. It is now immediate that $\rho(P_\lambda) \in \mathbb{Z}^p$ are linearly independent (over $\mathbb{Q}$) for $0 \leq \lambda \leq p-1$. Hence, ρ is a rational isomorphism. $\square$

Recall that $\mathbb{P}(\underline{s}l_2) \simeq \mathbb{P}^1$ (see Ex. 5.7), so that $K_0(\mathbb{P}(\underline{s}l_2)) \simeq \mathbb{Z}^{\oplus 2}$.

Let $\mathrm{Vect}(\mathbb{P}^1)$ denote the monoid of isomorphism classes of algebraic vector bundles on $\mathbb{P}^1$. We define

$$\gamma : \mathrm{Vect}(\mathbb{P}^1) \ \to \ \mathbb{Z}$$

by sending $E \simeq \bigoplus_n \mathcal{O}_{\mathbb{P}^1}(n)^{m_n}$ to $\dim \Gamma(\mathbb{P}^1, E) = \sum_n m_n(n+1)$. We shall use the following property of γ in the proof of Proposition 6.6

Lemma 6.5. *The map* $\gamma : \mathrm{Vect}(\mathbb{P}^1) \ \to \ \mathbb{Z}$ *extends to a homomorphism*

$$\gamma : K_0(\mathbb{P}^1) \ \to \ \mathbb{Z},$$

which fits into a commutative diagram:

$$\begin{tikzcd} K_0(u(sl_2)) \arrow[r] \arrow[dr, "\rho_j"'] & K_0(\mathbb{P}^1) \arrow[d, "\gamma"] \\ & \mathbb{Z} \end{tikzcd}$$

for any j, $1 \leq j \leq p-1$. *The top arrow is defined by* $[P] \mapsto [\mathrm{Ker}\{\widetilde{\Theta}^j_{\underline{s}l_2}, \mathcal{O}_{\mathbb{P}^1} \otimes P\}]$.

Proof. Clearly, $\bigoplus_n \mathcal{O}_{\mathbb{P}^1}(n)^{m_n} \mapsto \sum_n m_n(n+1)$ extends to $Vect(\mathbb{P}^1)^+$, the Grothendieck group of the group completion of the monoid $Vect(\mathbb{P}^1)$. We must show that γ so defined on $Vect^+(\mathbb{P}^1)$ factors through $K_0(\mathbb{P}^1)$. For any short sequence

$$0 \to \ E_1 \ \to \ E_2 \ \to \ E_3 \ \to 0 \tag{6.5.1}$$

in $Vect(\mathbb{P}^1)^+$, there exists some positive integer ℓ such that

$$0 \to E_1(\ell) \to E_2(\ell) \to E_3(\ell) \to 0 \tag{6.5.2}$$

is a short exact sequence in $Vect(\mathbb{P}^1)^+$ with $H^1(\mathbb{P}^1, E_1(\ell)) = 0$. Thus, γ is additive on (6.5.2). On the other hand, our explicit formula for γ implies that γ applied to $E \simeq \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(n)^{m_n}$ satisfies $\gamma(E(\ell)) = \gamma(E) + \sum_n m_n \cdot \ell$, so that the additivity of γ on (6.5.2) implies the additivity of γ on (6.5.1).

Since $\rho_j([P]) = \dim \Gamma(\mathbb{P}^1, \operatorname{Ker}\{\widetilde{\Theta}^j_{\underline{s}l_2}, \mathcal{O}_{\mathbb{P}^1} \otimes P\})$ by definition, the commutativity of the diagram is immediate. $\square$

Let X be an algebraic variety, and denote by $\widetilde{K}_0(X)$ the kernel of the rank function:

$$\widetilde{K}_0(X) = \operatorname{Ker}\{\operatorname{rk} : K_0(X) \to \mathbb{Z}\}.$$

Then $\widetilde{K}_0(X)$ splits off as a direct summand of $K_0(X)$ via the map

$$\widetilde{\operatorname{pr}} : K_0(X) \to \widetilde{K}_0(X), \quad [E] \mapsto [E] - (\operatorname{rk} E)[\mathcal{O}_X].$$

Let G be an infinitesimal group scheme. We denote by

$$\kappa_{G,j} : K_0(kG) \longrightarrow K_0(\mathbb{P}(\mathrm{G})), \quad \kappa_G \equiv \kappa_{G_1} \tag{6.5.3}$$

the map defined by sending a class of a finite-dimensional projective kG-module P to the class of a vector bundle $\operatorname{Ker}\{\widetilde{\Theta}^j_G, \mathcal{O}_{\mathbb{P}(\mathrm{G})} \otimes P\}$:

$$[P] \mapsto [\operatorname{Ker}\{\widetilde{\Theta}^j_G, \mathcal{O}_{\mathbb{P}(\mathrm{G})} \otimes P\}].$$

Let

$$\overline{\kappa}_{G,j} : K_0(kG) \longrightarrow \widetilde{K}_0(\mathbb{P}(\mathrm{G}))$$

be the composition

$$\overline{\kappa}_{G,j} = \widetilde{\operatorname{pr}} \circ \kappa_{G,j} : K_0(kG) \longrightarrow K_0(\mathbb{P}(\mathrm{G})) \longrightarrow \widetilde{K}_0(\mathbb{P}(\mathrm{G})).$$

We shall often supress the index G in $\kappa_{G,j}$ when the group scheme G is clear from the content.

Finally, we define a map

$$\nu : K_0(kG) \longrightarrow \widetilde{K}_0(\mathbb{P}(\mathrm{G}))^{p-1} \oplus \mathbb{Z} \tag{6.5.4}$$

via

$$K_0(kG) \xrightarrow{(\overline{\kappa}_1,\ldots,\overline{\kappa}_{p-1},\,\dim)} \widetilde{K}_0(\mathbb{P}(\mathrm{G}))^{p-1} \oplus \mathbb{Z}\,.$$

We now present a useful variant of Proposition 6.4.

Proposition 6.6. *The homomorphism*

$$\nu : K_0(\mathfrak{u}(sl_2)) \to \widetilde{K}_0(\mathbb{P}(\underline{s}l_2))^{p-1} \oplus Z$$

is a rational isomorphism.

Proof. Recall that $\mathbb{P}(\underline{s}l_2) \simeq \mathbb{P}^1$. Fix the isomorphism $f : \widetilde{K}_0(\mathbb{P}^1) \simeq \mathbb{Z}$ which sends $[\mathcal{O}_{\mathbb{P}^1}(1)^n - \mathcal{O}^n_{\mathbb{P}^1}]$ to n. Recall that the last component of the map ρ as defined in (6.3), ρ_p, is simply the dimension of a module. We have two commutative triangles

$$\begin{tikzcd} K_0(u(sl_2)) \arrow[r, "\kappa_j"] \arrow[dr, "\rho_j - \frac{j}{p}\rho_p"'] & K_0(\mathbb{P}^1) \arrow[d, "\gamma - \mathrm{rk}"] \\ & \mathbb{Z} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} K_0(\mathbb{P}^1) \arrow[r, "\widetilde{\mathrm{pr}}"] \arrow[d, "\gamma - \mathrm{rk}"'] & \widetilde{K}_0(\mathbb{P}^1) \arrow[dl, "f"] \\ \mathbb{Z} & \end{tikzcd}$$

where $\mathcal{P} = \mathcal{O}_{\mathbb{P}^1} \otimes P$, and γ as in Lemma 6.5. The commutativity of the first diagram follows from Lemma 6.5 and the fact that $\mathrm{rk}\,\mathrm{Ker}\{\Theta^j_{\underline{s}l_2}, \mathcal{P}\} = \frac{\dim P}{p} j = \frac{j}{p}\rho_p(P)$. To check that the second diagram commutes, it suffices to check on generators $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(1)$ of $K_0(\mathbb{P}^1)$. We compute $f(\widetilde{\mathrm{pr}}(\mathcal{O}_{\mathbb{P}^1})) = f(0) = 0$, and $f(\widetilde{\mathrm{pr}}(\mathcal{O}_{\mathbb{P}^1}(1))) = f(\mathcal{O}_{\mathbb{P}^1}(1) - \mathcal{O}_{\mathbb{P}^1}) = 1 = \dim \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) - \mathrm{rk}\,\mathcal{O}_{\mathbb{P}^1}(1) = (\gamma - \mathrm{rk})[\mathcal{O}_{\mathbb{P}^1}(1)]$.

Concatinating the two diagrams above, we obtain the following commutative diagram for any j, $1 \leq j \leq p-1$,

(6.6.1)
$$\begin{tikzcd} K_0(u(sl_2)) \arrow[r, "\overline{\kappa}_j"] \arrow[dr, "\rho_j - \frac{j}{p}\rho_p"'] & \widetilde{K}_0(\mathbb{P}^1) \arrow[d, "\sim"] \\ & \mathbb{Z}. \end{tikzcd}$$

The matrix of integer vectors

$$\langle(\rho_1 - \frac{1}{p}\rho_p)(P_\lambda), \ldots, (\rho_{p-1} - \frac{p-1}{p}\rho_p)(P_\lambda), \rho_p(P_\lambda)\rangle,$$

$0 \leq \lambda \leq p-1$, is obtained via column operations (subtracting multiples of the last column) from the matrix

$$\rho(P_\lambda) = \langle \rho_1(P_\lambda), \ldots, \rho_p(P_\lambda)\rangle$$

and, hence, has the same rank. Since $\rho(P_\lambda)$, $0 \leq \lambda \leq p-1$, are linearly independent over $\mathbb{Q}$ by Proposition 6.4, we conclude that $\langle(\rho_1 - \frac{1}{p}\rho_p)(P_\lambda), \ldots, (\rho_{p-1} - \frac{p-1}{p}\rho_p)(P_\lambda), \rho_p(P_\lambda)\rangle$ are also linearly independent. The commutativity of the diagram (6.6.1) implies that $\langle f \circ \overline{\kappa}_1(P_\lambda), \ldots, f \circ \overline{\kappa}_{p-1}(P_\lambda), \dim_k(P_\lambda)\rangle$, $0 \leq \lambda \leq p-1$, are linearly independent. Since $f : \widetilde{K}^0(\mathbb{P}^1) \to \mathbb{Z}$ is an isomorphism, we conclude that $\nu : K_0(kG) \to \widetilde{K}_0(\mathbb{P}^1)^{p-1} \oplus \mathbb{Z}$ is a rational isomorphism. $\square$

Recall that if $H \subset G$ is a subgroup scheme of a finite group scheme G, then kG is free as a kH module [21, 2.4]. The next proposition enables us to conclude that the rank of ν_H is a lower bound for the rank of ν_G.

Proposition 6.7. *Let G be an infinitesimal group scheme, $i : H \subset G$ a subgroup scheme, and d the rank of kG as a free kH–module. Let $f : \mathbb{P}(H) \hookrightarrow \mathbb{P}(\mathrm{G})$ be the closed embedding of projective varieties induced by the embedding of group schemes. Consider the maps of the following square*

(6.7.1)
$$\begin{tikzcd} K_0(kH) \arrow[r, "\nu_H"] \arrow[d, "i_*"] & \widetilde{K}_0(\mathbb{P}(H))^{p-1} \oplus \mathbb{Z} \\ K_0(kG) \arrow[r, "\nu_G"] & \widetilde{K}_0(\mathbb{P}(\mathrm{G}))^{\times p-1} \oplus \mathbb{Z}. \arrow[u, "(f^*)^{\times(p-1)} \oplus \mathrm{id}"'] \end{tikzcd}$$

Here, $i_*: K_0(kH) \to K_0(kG)$ *is induced by the tensor induction functor*

$$kG \otimes_{kH} - : kH - \mathrm{mod} \longrightarrow kG - \mathrm{mod}$$

which takes projective modules to projective modules, and $f^*: K_0(\mathbb{P}(\mathrm{G})) \to K_0(\mathbb{P}(H))$ *is induced by the pull-back functor*

$$f^*: (\mathcal{O}_{\mathbb{P}(\mathrm{G})} - \mathrm{mod}) \longrightarrow (\mathcal{O}_{\mathbb{P}(H)} - \mathrm{mod}).$$

Then $((f^*)^{\times(p-1)} \oplus \mathrm{id}) \circ \nu_G \circ i_* \ = d \cdot \nu_H$.

Proof. Fix any j, $1 \le j \le p$. We first show that for the following diagram

$$\begin{array}{ccc} K_0(kH) & \xrightarrow{\kappa_{H,j}} & K_0(\mathbb{P}(H)) \\ \downarrow{\scriptstyle i_*} & & \uparrow{\scriptstyle f^*} \\ K_0(kG) & \xrightarrow{\kappa_{G,j}} & K_0(\mathbb{P}(\mathrm{G})), \end{array} \tag{6.7.2}$$

we have a relation $f^* \circ \kappa_{G,j} \circ i_* = d \cdot \kappa_{H,j}$.

By Proposition 5.3,

$$f^*(\mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{O}_{\mathbb{P}(\mathrm{G})} \otimes M\}) \simeq \mathrm{Ker}\{\widetilde{\Theta}^j_H, \mathcal{O}_{\mathbb{P}(H)} \otimes M_{|H}\} \tag{6.7.3}$$

for any G–module M. Let P be a finite dimensional projective H-mod. Applying (6.7.3) to the G-module $kG \otimes_{kH} P$, we get

$$f^*(\mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{O}_{\mathbb{P}(\mathrm{G})} \otimes kG \otimes_{kH} P\}) \simeq \mathrm{Ker}\{\widetilde{\Theta}^j_H, \mathcal{O}_{\mathbb{P}(H)} \otimes (kG \otimes_{kH} P)_{|H}\} \simeq$$
$$\mathrm{Ker}\{\widetilde{\Theta}^j_H, \mathcal{O}_{\mathbb{P}(H)} \otimes P^{\oplus d}\} \simeq \mathrm{Ker}\{\widetilde{\Theta}^j_H, \mathcal{O}_{\mathbb{P}(H)} \otimes P\}^{\oplus d}.$$

Hence, $f^* \circ \kappa_{G,j} \circ i_* = d \cdot \kappa_{H,j}$.

The diagrams

$$\begin{array}{ccc} K_0(\mathbb{P}(H)) & \xrightarrow{\widetilde{\mathrm{pr}}} & \widetilde{K}_0(\mathbb{P}(H)) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ K_0(\mathbb{P}(\mathrm{G})) & \xrightarrow{\widetilde{\mathrm{pr}}} & \widetilde{K}_0(\mathbb{P}(\mathrm{G})) \end{array} \quad \text{and} \quad \begin{array}{ccc} K_0(\mathbb{P}(H)) & \xrightarrow{\mathrm{rk}} & \mathbb{Z} \\ {\scriptstyle f^*}\uparrow & & \| \\ K_0(\mathbb{P}(\mathrm{G})) & \xrightarrow{\mathrm{rk}} & \mathbb{Z} \end{array} \tag{6.7.4}$$

are clearly commutative. The statement now follows by observing that concatinating the square (6.7.2) for $1 \le j \le p-1$ with the first diagram of 6.7.4 gives the j-th component of the square (6.7.1), and concatinating (6.7.2) with the second diagram of (6.7.4) gives the last component of (6.7.1). □

To extend our computation we need the following linear algebra lemma.

Lemma 6.8. *Let* V, W *be finite-dimensional vector spaces, and let* $\alpha \in \mathrm{End}(V)$, $\beta\ in\, \mathrm{End}(W)$ *be nilpotent operators on* V *and* W *respectively. Let* $\gamma: V \otimes W \to V \otimes W$ *be the operator defined as* $\gamma(v \otimes w) = \alpha(v) \otimes \beta(w)$. *Then the following sequence is exact:*

$$0 \to \mathrm{Ker}\,\alpha \otimes \mathrm{Ker}\,\beta \to \mathrm{Ker}\,\alpha \otimes W \oplus V \otimes \mathrm{Ker}\,\beta \to \mathrm{Ker}\,\gamma \to 0. \tag{6.8.1}$$

Proof. Let $V = \bigoplus V_i$, $W = \bigoplus W_j$ be the decompositions into generalized eigenspaces with respect to the operators α, β. It sufficed to prove the statement for $V = V_i$, $W = W_j$. Hence, we may assume that V has a basis $\langle e, \alpha e, \dots, \alpha^{i-1} e\rangle$ with $\dim V = i$, and W has a basis $\langle f, \beta f, \dots, \beta^{j-1} f\rangle$ with $\dim W = j$. Suppose $v \otimes w = (\sum a_n \alpha^n e) \otimes (\sum b_m \beta^m f) \in \mathrm{Ker}\,\gamma$. Then $\alpha(v) \otimes \beta(w) = (\sum a_n \alpha^{n+1} e) \otimes$

$(\sum b_m \beta^{m+1} f) = 0$. Since $\langle \alpha^n e \otimes \beta^m f \rangle_{0 \le n \le i-1, 0 \le m \le j-1}$ is a basis of $V \otimes W$, we conclude that $v \otimes w = a_{n-1}\alpha^{n-1} e \otimes w' + v' \otimes b_{m-1} f$ for some $v' \in V, w' \in W$. Hence, $\operatorname{Ker}\gamma = \operatorname{Ker}\alpha \otimes W + V \otimes \operatorname{Ker}\beta$. □

Corollary 6.9. *Let G_1, G_2 be infinitesimal groups scheme, and let M_1, M_2 be representations of G_1, G_2 respectively of constant rank. Then the $G_1 \times G_2$-module $M_1 \otimes M_2$ has constant rank.*

Proof. Let $x \in V(G_1)$, $y \in V(G_2)$, and let $\mu_x : \mathbb{G}_{a(r),k(x)} \to G_{1,k(x)}$, $\mu_y : \mathbb{G}_{a(r),k(y)} \to G_{2,k(y)}$ be the corresponding one-parameter subgroups. Then the nilpotent operator

$$\theta_{(x,y)} : M_1 \otimes M_2 \otimes k_{(x,y)} \to M_1 \otimes M_2 \otimes k_{(x,y)}$$

corresponding to the one-parameter subgroup

$$\mu_{(x,y)} \mathbb{G}_{a(r),k(x,y)} \xrightarrow{\Delta} \mathbb{G}_{a(r),k(x,y)} \times \mathbb{G}_{a(r),k(x,y)} \xrightarrow{\mu_x \times \mu_y} G_{1,k(x,y)} \times G_{2,k(x,y)}$$

is given by $\theta_{(x,y)} = \theta_{x,k(x,y)} \otimes \theta_{y,k(x,y)} : (M_1 \otimes k(x,y)) \otimes_{k(x,y)} (M_2 \otimes k(x,y)) \to (M_1 \otimes k(x,y)) \otimes_{k(x,y)} (M_2 \otimes k(x,y))$. Lemma 6.8 now implies that the rank of $\theta_{(x,y)}$ is expressed in terms of the ranks of θ_x, θ_y and the dimensions of M_1, M_2. Hence, if M_1, M_2 have constant rank, so does $M_1 \otimes M_2$. □

Let G_1, G_2 be group schemes, and Q_1, Q_2 be modules for G_1, G_2 respectively. We consider $Q_1 \otimes Q_2$ as a $G_1 \times G_2$-module with respect to the standard component-wise action of $G_1 \times G_2$. For varieties X_1, X_2 and sheaves $\mathcal{F}_1$, $\mathcal{F}_2$, we denote by $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ the sheaf on $X_1 \times X_2$ which is the external tensor product of $\mathcal{F}_1$ and $\mathcal{F}_2$.

Proposition 6.10. *Let G_1 and G_2 be infinitesimal group schemes, let $G = G_1 \times G_2$, and let*

$$i : \mathbb{P}(G_1) \times \mathbb{P}(G_2) \longrightarrow \mathbb{P}(G_1 \times G_2)$$

be the map induced by the bihomogeneous isomorphism $V(G_1) \times V(G_2) \simeq V(G_1 \times G_2)$. Let Q_i be a projective G_i-module, and let $\mathcal{Q}_i = \mathcal{O}_{\mathbb{P}(G_i)} \otimes Q_i$ be the corresponding free sheaf on $\mathbb{P}(G_i)$, $i = 1, 2$. Let $\mathcal{Q}_{1,2} = \mathcal{O}_{\mathbb{P}(G_1 \times G_2)} \otimes (Q_1 \otimes Q_2)$ be the free sheaf on $\mathbb{P}(G_1 \times G_2)$ corresponding to $Q_1 \otimes Q_2$. Then we have a short exact sequence of algebraic vector bundles on $\mathbb{P}(G_1) \times \mathbb{P}(G_2)$:

$$(6.10.1) \quad 0 \longrightarrow \operatorname{Ker}\{\widetilde{\Theta}_{G_1}, \mathcal{Q}_1\} \boxtimes \operatorname{Ker}\{\widetilde{\Theta}_{G_2}, \mathcal{Q}_2\} \longrightarrow (\operatorname{Ker}\{\widetilde{\Theta}_{G_1}, \mathcal{Q}_1\} \boxtimes \mathcal{Q}_2) \oplus$$

$$(\mathcal{Q}_1 \boxtimes \operatorname{Ker}\{\widetilde{\Theta}_{G_1}, \mathcal{Q}_1\}) \longrightarrow i^* \operatorname{Ker}\{\widetilde{\Theta}_{G_1 \times G_2}, \mathcal{Q}_{1,2}\} \longrightarrow 0.$$

Proof. We have a commutative diagram of sheaves on $\mathbb{P}(G_1) \times \mathbb{P}(G_2)$:

$$\begin{array}{ccc} \mathcal{Q}_1 \boxtimes \mathcal{Q}_2 & \xrightarrow{\sim} & i^*(\mathcal{Q}_{1,2}) \\ \downarrow{\scriptstyle \widetilde{\Theta}_{G_1} \boxtimes \widetilde{\Theta}_{G_2}} & & \downarrow{\scriptstyle i^*\widetilde{\Theta}_{G_1 \times G_2}} \\ \mathcal{Q}_1 \boxtimes \mathcal{Q}_2 & \xrightarrow{\sim} & i^*(\mathcal{Q}_{1,2}). \end{array}$$

Hence,

$$i^* \operatorname{Ker}\{\widetilde{\Theta}_{G_1 \times G_2}, \mathcal{Q}_{1,2}\} = \operatorname{Ker}\{i^*\widetilde{\Theta}_{G_1 \times G_2}, i^*\mathcal{Q}_{1,2}\} = \operatorname{Ker}\{\widetilde{\Theta}_{G_1} \boxtimes \widetilde{\Theta}_{G_2}, \mathcal{Q}_1 \boxtimes \mathcal{Q}_2\}.$$

The latter fits into a short exact sequence

$$0 \longrightarrow \operatorname{Ker}\{\widetilde{\Theta}_{G_1}, \mathcal{Q}_1\} \boxtimes \operatorname{Ker}\{\widetilde{\Theta}_{G_2}, \mathcal{Q}_2\} \longrightarrow (\operatorname{Ker}\{\widetilde{\Theta}_{G_1}, \mathcal{Q}_1\} \boxtimes \mathcal{Q}_2) \oplus$$

$$(\mathcal{Q}_1 \boxtimes \operatorname{Ker}\{\widetilde{\Theta}_{G_1}, \mathcal{Q}_1\}) \longrightarrow \operatorname{Ker}\{\widetilde{\Theta}_{G_1} \boxtimes \widetilde{\Theta}_{G_2}, \mathcal{Q}_1 \boxtimes \mathcal{Q}_2\} \longrightarrow 0$$

since this sequence is short exact at every fiber by Lemma 6.8 and Theorem 4.12. □

Corollary 6.11. *Let G_1 and G_2 be infinitesimal group schemes, let $G = G_1 \times G_2$, and let*

$$i : \mathbb{P}(G_1) \times \mathbb{P}(G_2) \longrightarrow \mathbb{P}(G_1 \times G_2)$$

be the map induced by the bihomogeneous isomorphism $V(G_1) \times V(G_2) \simeq V(G_1 \times G_2)$. Let $Q_i \in K_0(G_i)$, and let $\mathcal{O}_i = \mathcal{O}_{\mathbb{P}(G_i)}$, $i = 1, 2$. Assume that $K_0(\mathbb{P}(G_1) \times \mathbb{P}(G_2)) = K_0(\mathbb{P}(G_1)) \otimes K_0(\mathbb{P}(G_2)))$. Let $\mathcal{O}_i = \mathcal{O}_{\mathbb{P}(G_i)}$. Then

$$(i^* \circ \kappa_{G_1 \times G_2})(Q_1 \otimes Q_2) =$$

$$(\operatorname{rk} Q_2)\kappa_{G_1}(Q_1) \otimes [\mathcal{O}_2] + (\operatorname{rk} Q_1)[\mathcal{O}_1] \otimes \kappa_{G_2}(Q_2) - \kappa_{G_1}(Q_1) \otimes \kappa_{G_2}(Q_2).$$

Moreover, let $G_1, \ldots, G_r$ be infinitesimal group schemes, and assume

$$K_0(\mathbb{P}(G_1) \times \cdots \times \mathbb{P}(G_r)) = K_0(\mathbb{P}(G_1)) \otimes \cdots \otimes K_0(\mathbb{P}(G_r)).$$

Let $i : \mathbb{P}(G_1) \times \cdots \times \mathbb{P}(G_r) \hookrightarrow \mathbb{P}(G_1 \times \cdots \times G_r)$ be the natural embedding. Let $\mathcal{O}_i = \mathcal{O}_{\mathbb{P}(G_i)}$, and let $Q_i \in K_0(G_i)$, $i = 1, \ldots, r$. Let $q_i = \operatorname{rk} Q_i$. Then

$$(i^* \circ \kappa_{G_1 \times \cdots \times G_r})(Q_1 \otimes \cdots \otimes Q_r) = \sum_i \frac{q_1 \ldots q_r}{q_i} [\mathcal{O}_1] \otimes [\mathcal{O}_2] \otimes \ldots \otimes \kappa_{G_i}(Q_i) \otimes \ldots \otimes [\mathcal{O}_r] -$$

$$\sum_{1 \le i < j \le r} \frac{q_1 \ldots q_r}{q_i q_j} [\mathcal{O}_1] \otimes [\mathcal{O}_2] \otimes \ldots \otimes \kappa_{G_i}(Q_i) \otimes \ldots \otimes \kappa_{G_j}(Q_j) \otimes \ldots \otimes [\mathcal{O}_r] \ldots \pm \bigotimes_{i=1}^r \kappa_{G_i}(Q_i)$$

in $K_0(\mathbb{P}(G_1) \times \cdots \times \mathbb{P}(G_r))$.

Proof. Since all operations in Proposition 6.10 commute with direct sums, the statement for $r = 2$ is a direct consequence of that Proposition. The second claim follows by induction. □

In conjunction with Proposition 6.7, Proposition 6.12 provides a lower bound for the rank of $K_0(\mathbb{P}(G))$ whenever $\underline{s}l_2^{\times r}$ is a subgroup of the infinitesimal group scheme G.

Recall that $\mathbb{P}(\underline{s}l_2) \simeq \mathbb{P}^1$, and $\mathbb{P}(\underline{s}l_2^{\oplus r}) \simeq \mathbb{P}^{2r-1}$. We have a natural embedding $i : (\mathbb{P}^1)^{\times r} \to \mathbb{P}^{2r-1}$ and the induced map $i^* : K_0(\mathbb{P}^{2r-1}) \to K_0(\mathbb{P}^1)^{\times r}$ has rank $r+1$. We identify the map $i^* : K_0(\mathbb{P}^{2r-1}) \to K_0((\mathbb{P}^1)^{\times r}) \simeq K_0(\mathbb{P}^1)^{\otimes r}$ as the map $\mathbb{Z}[t]/t^{2r} \to \bigotimes_{i=1}^r \mathbb{Z}[t_i]/t_i^2$ given by $t \mapsto \sum_i t_i$.

In the next Proposition we essentially investigate the following diagram:

$$\begin{tikzcd}
K_0(\underline{s}l_2^{\oplus r}) \arrow[r, "\kappa_{(\underline{s}l_2^{\oplus r})}"] & K_0(\mathbb{P}^{2r-1}) \arrow[d, "i^*"] \\
K_0(\underline{s}l_2)^{\otimes r} \arrow[u, "\otimes"] \arrow[r, "(\kappa_{\underline{s}l_2})^{\otimes r}"] & K_0(\mathbb{P}^1)^{\otimes r}.
\end{tikzcd}$$

It is important to keep in mind that although all entries are commutative rings, the diagram does not respect the product structure since the map $\kappa : K_0(kG) \to K_0(\mathbb{P}(G))$ is not multiplicative.

Proposition 6.12. *Let $\mathfrak{h} = sl_2^{\oplus r}$ for some $r \geq 1$ and let $H = \underline{\mathfrak{h}}$. Then the composition*

$$i^* \circ \kappa_H : K_0(kH) \longrightarrow K_0(\mathbb{P}^{2r-1}) \longrightarrow K_0(\mathbb{P}^1)^{\otimes r}$$

has rank at least $r+1$.

Proof. We readily verify that the multi-homogeneous map $V(\underline{sl}_2)^{\times r} \to V(\underline{\mathfrak{h}})$ induces the natural embedding $i : (\mathbb{P}^1)^{\times r} \to \mathbb{P}^{2r-1}$. Let $\epsilon_j : \mathbb{P}^1 \to (\mathbb{P}^1)^{\times r}$ be the morphism whose composition with the i-th projection sends $\mathbb{P}^1$ to the point at infinity for $i \neq j$ and whose composition with the j-th projection is the identity.

By Proposition 6.6, the map $(\widetilde{\mathrm{pr}} \circ \kappa_{sl_2}, \dim) : K_0(u(sl_2)) \to \widetilde{K}_0(\mathbb{P}^1) \otimes \mathbb{Z}$ is rationally surjective. Hence, there exists a class $Q \in K_0(u(sl_2))$ such that $\widetilde{\mathrm{pr}} \circ \kappa_{sl_2}(Q) = 0$ but $\dim(Q) \neq 0$. In other words, Q is a non-trivial class satisfying the condition $\widetilde{\mathrm{pr}} \circ \kappa_{sl_2}(Q) = 0$. Moreover, for any projective module P we have an equality $\dim P = p \cdot \mathrm{rk}(\kappa_{sl_2}(P))$. Hence, Q satisfies the condition $\dim Q \neq \mathrm{rk}(\kappa_{sl_2}(Q))$.

Let $S = S_{p-1}$ be the Steinberg module. By Example 5.7, $\kappa_{sl_2}(S) = [\mathcal{O}(1-p)] \in K_0(\mathbb{P}^1)$. Let $s_i = Q \otimes \ldots \otimes S \otimes \ldots \otimes Q$ be the class in $K_0(kH)$, with S at the ith place, for $1 \leq i \leq r$. We shall show that the class $s_i \in K_0(kH)$ satisfies the property

$$(\widetilde{\mathrm{pr}} \circ \epsilon_j^*)(i^* \circ \kappa_H)(s_i) \neq 0 \text{ iff } i = j \tag{6.12.1}$$

as a class in $\widetilde{K}_0(u(sl_2))$.

Let $r = 2$ so that $H = \underline{sl}_2^{\oplus 2}$, and consider the short exact sequence (6.10.1) applied to $G_1 = G_2 = \underline{sl}_2$. Since $K_0(\mathbb{P}^1 \times \mathbb{P}^1) = K_0(\mathbb{P}^1) \otimes K_0(\mathbb{P}^1)$, we can apply Corollary 6.11 to get a relation in $K_0(\mathbb{P}^1 \times \mathbb{P}^1)$. Let $q = \dim Q$, and consider $s_1 = Q \otimes S$. Corollary 6.11 implies

$$(i^* \circ \kappa_H)(Q \otimes S) = p \cdot \kappa_{sl_2}(Q) \otimes [\mathcal{O}_2] + q \cdot [\mathcal{O}_1] \otimes \kappa_{sl_2}(S) - \kappa_{sl_2}(Q) \otimes \kappa_{sl_2}(S). \tag{6.12.2}$$

Let $\epsilon_1 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the natural embedding of the first component. Then

$$\epsilon_1^* : K_0(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z}[t_1]/t_1^2 \otimes \mathbb{Z}[t_2]/t_2^2 \to K_0(\mathbb{P}^1) \simeq \mathbb{Z}[t_1]/t_1^2$$

is explicitly given by $\epsilon_1^*(t_1) = t_1, \epsilon_1^*(t_2) = 0$. Applying ϵ_1^* to (6.12.2), we get

$$\epsilon_1^* \circ (i^* \circ \kappa_H)(Q \otimes S) = p \cdot \epsilon_1^*(\kappa_{sl_2}(Q))\epsilon_1^*([\mathcal{O}_2]) + q \cdot \epsilon_1^*([\mathcal{O}_1])\epsilon_1^*(\kappa_{sl_2}(S)) - \epsilon_1^*(\kappa_{sl_2}(Q))\epsilon_1^*(\kappa_{sl_2}(S)) =$$

$$p \cdot \kappa_{sl_2}(Q) + q \cdot \mathrm{rk}(\kappa_{sl_2}(S)) - \kappa_{sl_2}(Q) \cdot \mathrm{rk}(\kappa_{sl_2}(S)) = p \cdot \kappa_{sl_2}(Q) + q[\mathcal{O}] - \kappa_{sl_2}(Q).$$

where the last equality uses the fact that $\mathrm{rk}(\kappa_{sl_2}(S)) = 1$ (see Example 5.7). The condition on the class $[Q]$ implies that the projection map $\widetilde{\mathrm{pr}} : K_0(u(sl_2)) \to \widetilde{K}_0(u(sl_2))$ sends this class to zero.

We now compute the effect of ϵ_2^* on $(i^* \circ \kappa_H)(Q \otimes S)$:

$$\begin{aligned} \epsilon_2^* \circ (i^* \circ \kappa_H)(Q \otimes S) &= p \cdot \epsilon_2^*(\kappa_{sl_2}(Q))\epsilon_2^*([\mathcal{O}_2]) + q \cdot \epsilon_2^*([\mathcal{O}_1])\epsilon_2^*(\kappa_{sl_2}(S)) - \epsilon_2^*(\kappa_{sl_2}(Q))\epsilon_2^*(\kappa_{sl_2}(S)) \\ &= p \cdot \mathrm{rk}(\kappa_{sl_2}(Q)) + (q - \mathrm{rk}(\kappa_{sl_2}(Q)))\kappa_{sl_2}(S). \end{aligned}$$

Applying $\widetilde{\mathrm{pr}} : K_0(u(sl_2)) \to \widetilde{K}_0(u(sl_2))$, we get $(q - \mathrm{rk}(\kappa_{sl_2}(Q)))\kappa_{sl_2}(S) = (\mathrm{rk}\, Q - \mathrm{rk}(\kappa_{sl_2}(Q)))[\mathcal{O}(1-p)]$ which is non-zero since $\dim Q \neq \mathrm{rk}(\kappa_{sl_2}(Q))$.

For general r, we apply the second part of the Corollary 6.11 and argue exactly as in the case of $r = 2$ to conclude that $(\widetilde{\mathrm{pr}} \circ \epsilon_j^*) \circ (i^* \circ \kappa_H)(s_i) = 0$ for $i \neq j$. For $i = j$, we get the formula

$$(\widetilde{\mathrm{pr}} \circ \epsilon_j^*) \circ (i^* \circ \kappa_H)(s_j) = [q^{r-1} - q^{r-2}\,\mathrm{rk}(\kappa_{sl_2}(Q)) + q^{r-3}\,\mathrm{rk}(\kappa_{sl_2}(Q))^2 - \ldots$$

$$\pm \operatorname{rk}(\kappa_{sl_2}(Q))^{r-1})\kappa_{sl_2}(S) = \frac{q^r - \operatorname{rk}(\kappa_{sl_2}(Q))^r}{q - \operatorname{rk}(\kappa_{sl_2}(Q))}[\mathcal{O}(1-p)] \neq 0$$

since $q = \dim Q \neq \operatorname{rk}(\kappa_{sl_2}(Q))$.

Hence, the property (6.12.1) holds. This immediately implies that the image of $\pi^* \circ \kappa_H$ contains a non-zero multiple of t_i for all i, $1 \leq i \leq r$, so that the rank of the image has rank at least r.

Let $s_0 = Q \otimes \ldots \otimes Q$. Arguing as above, we conclude that

$$(\widetilde{\operatorname{pr}} \circ \epsilon_j^*)(i^* \circ \kappa_H)(s_0) = 0 \text{ for all } j$$

as an element in $\widetilde{K}_0(u(sl_2))$. On the other hand, applying the second part of Corollary 6.11, we compute

$$\operatorname{rk}[(i^* \circ \kappa)(s_0)] = \frac{q^r - \operatorname{rk}(\kappa_{sl_2}(Q))^r}{q - \operatorname{rk}(\kappa_{sl_2}(Q))}$$

which is non-zero by our assumption on Q. Hence, $i^* \circ \kappa(s_0)$ is linearly independent of $\{i^* \circ \kappa(s_1), \ldots, i^* \circ \kappa(s_r)\}$. Hence, the rank of the image of $i^* \circ \kappa$ is at least $r+1$. $\square$

Let $E = \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$, so that $kE \simeq k[x,y]/(x^p, y^p)$. Let X_n be $(2n-1)$–dimensional "zig-zag" module. Pictorially, we represent X_n by the following diagram:

$$\begin{tikzcd}
\overset{\langle v_1\rangle}{\bullet} \arrow[dr, "y"] & & \overset{\langle v_2\rangle}{\bullet} \arrow[dl, "x"'] \arrow[dr, "y"] & & \cdots & & \overset{\langle v_n\rangle}{\bullet} \arrow[dl, "x"'] \\
& \bullet & & \bullet & \cdots & \bullet &
\end{tikzcd}$$

It is straightforward to check that X_n has constant Jordan type $(n-1)[2]+[1]$ (see [6, §2]). We proceed to prove that for any integer m we can obtain the line bundle $\mathcal{O}_{\mathbb{P}^1}(m)$ on $\mathbb{P}(E) = \mathbb{P}^1$ by applying our constructions to some X_n or its linear dual $X_n^{\#}$.

There is an inclusion $\operatorname{Im}\{\widetilde{\Theta}_E, \mathcal{O}_{\mathbb{P}^1} \otimes X_n\} \subset \operatorname{Ker}\{\widetilde{\Theta}_E, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes X_n\}$. Hence, we may define

$$\mathcal{X}_n := \operatorname{Ker}\{\widetilde{\Theta}_E, \mathcal{O}_{\mathbb{P}^1} \otimes X_n\} / \operatorname{Im}\{\widetilde{\Theta}_E, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes X_n\}.$$

Arguing as in the proof of Proposition 5.14, one verifies that $\mathcal{X}_n$ is locally free with the fiber at a point $t \in \mathbb{P}^1$ isomorphic to the 1-dimensional vector space $\frac{\operatorname{Ker}\{\theta_t : X_{n,k(t)} \to X_{n,k(t)}\}}{\operatorname{Im}\{\theta_t : X_{n,k(t)} \to X_{n,k(t)}\}}$. Hence, $\mathcal{X}_n$ is a line bundle. The linear dual $X_n^{\#}$ of X_n has diagram:

$$\begin{tikzcd}
& \bullet \arrow[dl, "x"'] \arrow[dr, "y"] & & \bullet \arrow[dl, "x"'] & \cdots & \bullet \arrow[dr, "y"] & \\
\underset{\langle w_1\rangle}{\bullet} & & \underset{\langle w_2\rangle}{\bullet} & & \cdots & & \underset{\langle w_n\rangle}{\bullet}
\end{tikzcd}$$

Define

$$\mathcal{Y}_n := \operatorname{Ker}\{\widetilde{\Theta}_E, \mathcal{O}_{\mathbb{P}^1}(1) \otimes X_n^{\#}\} / \operatorname{Im}\{\widetilde{\Theta}_E, \mathcal{O}_{\mathbb{P}^1} \otimes X_n^{\#}\}.$$

One readily verifies that $\mathcal{Y}_n$ is dual to $\mathcal{X}_n$ as a vector bundle on $\mathbb{P}^1$.

Proposition 6.13. $\mathcal{X}_n \simeq \mathcal{O}_{\mathbb{P}^1}(-n)$, $\mathcal{Y}_n \simeq \mathcal{O}_{\mathbb{P}^1}(n)$.

Proof. Let $k[s,t] = k[\mathbb{A}^2] \simeq k[V(E)]$. The universal p-nilpotent operator $\Theta_E \in k[s,t] \otimes k[x,y]/(x^p, y^p)$ is given by

$$\Theta_E = sx + ty,$$

(see, for example, Ex. 2.5(1)). We identify the graded $k[s,t]$-module $\operatorname{Ker}\{\Theta_E, k[s,t] \otimes X_n\}/\operatorname{Im}\{\Theta_E, k[s,t] \otimes X_n\}$, thereby determining to the vector bundle $\mathcal{X}_N$. It is easy to see that $\operatorname{Im}\{\Theta_E, k[s,t] \otimes X_n\}$ is generated by $\langle v_1, \dots, v_n\rangle$ as a $k[s,t]$–module and $\operatorname{Ker}\{\Theta_E, k[s,t] \otimes X_n\}$ is generated by $\langle v_1, \dots, v_n, s^n v_1 + s^{n-1} t v_{n-1} + \cdots t^n v_n\rangle$ as a $k[s,t]$–module. Hence, $\operatorname{Ker}\{\Theta_E, k[s,t] \otimes X_n\}/\operatorname{Im}\{\Theta_E, k[s,t] \otimes X_n\}$ is generated by $\langle s^n v_1 + s^{n-1} t v_{n-1} + \cdots t^n v_n\rangle$ as a $k[s,t]$-module. Since the generator is in degree n, we conclude that the corresponding locally free sheaf of rank 1 is $\mathcal{O}_{\mathbb{P}^1}(-n)$.

We now compute $\mathcal{Y}_n(-1) = \operatorname{Ker}\{\widetilde{\Theta}_E, \mathcal{O}_{\mathbb{P}^1} \otimes X_n^{\#}\}/\operatorname{Im}\{\widetilde{\Theta}_E, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes X_n^{\#}\}$. The graded $k[s,t]$-module $\operatorname{Ker}\{\Theta_E, k[s,t] \otimes X_n^{\#}\}$ is generated by $\langle w_1, \dots, w_n\rangle$ in degree 0, and $\operatorname{Im}\{\Theta_E, k[s,t](-1) \otimes X_n^{\#}\}$ is generated by $\langle sw_1 + tw_2, sw_2 + tw_3, \dots, sw_{n-1} + tw_n\rangle$, also in degree 0. Hence, on $U_0 = \mathbb{P}^1 - Z(s)$, the restriction of $\mathcal{Y}_n(-1)$ is generated by w_n, with $w_1 = (-\frac{t}{s})^{n-1} w_n$. We map $\mathcal{Y}_n(-1)(U_0)$ to $K = k(t/s)$, the residue field at the generic point of $\mathbb{P}^1$, by sending w_n to 1. The image of w_1 is $(-\frac{t}{s})^{n-1}$. On the other affine piece, $U_1 = \mathbb{P}^1 - Z(t)$, the restriction of $\mathcal{Y}_n(-1)$ is generated by w_1, with the relation $w_n = (-\frac{s}{t})^{n-1} w_1$. We map this to $K = k(s/t)$ by sending w_1 to $(-\frac{t}{s})^{n-1}$. Hence, the vector bundle is given by the Cartier divisor $(U_0, 1), (U_1, (-\frac{t}{s})^{n-1})$. This divisor is equivalent to the Cartier divisor $(U_0, 1), (U_1, (\frac{t}{s})^{n-1})$ which correspond to the line bundle $\mathcal{O}_{\mathbb{P}^1}(n-1)$. Hence, $\mathcal{Y}_n(-1) \simeq \mathcal{O}_{\mathbb{P}^1}(n-1)$. Therefore, $\mathcal{Y}_n \simeq \mathcal{O}_{\mathbb{P}^1}(n)$. □

In the next Proposition we calculate explicitly the line bundles corresponding to the syzygies of the trivial modules, $\Omega^n k$. For convenience, we use the notation $\mathcal{H}^{[1]}(M)$ for the bundle $\mathcal{M}^{[1]}$ associated to M as defined in (5.12).

Proposition 6.14. *Let $E = \mathbb{G}_{a(1)}^{\times r}$. Then*

$$\mathcal{H}^{[1]}(\Omega^n k) \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^{r-1}}(-\frac{np}{2}) & \text{if } n \text{ is even,} \\ \mathcal{O}_{\mathbb{P}^{r-1}}(-\frac{n+1}{2}p + 1) & \text{if } n \text{ is odd} \end{cases}$$

for p odd and

$$\mathcal{H}^{[1]}(\Omega^n k) \simeq \mathcal{O}_{\mathbb{P}^{r-1}}(-n)$$

for $p = 2$.

Proof. Let $r = 2$, and assume $n \geq 0$. As in the proof of Prop. 6.13, the universal operator $\Theta_E = sx + ty$ where $k[V(E)] = k[s,t]$. The structure of a minimal $kE \simeq k[x,y]/(x^p, y^p)$-projective resolution $P_\bullet \to k$ is well known [7], with $P_{n-1} = kG^{\times n}$. A set of generators $a_1, \dots, a_n$ for P_{n-1} can be chosen so that $\Omega^n(k)$ is the submodule generated by the elements

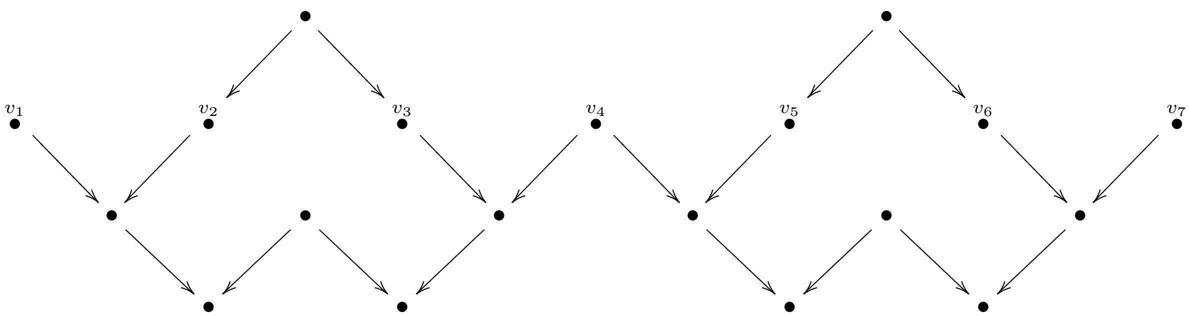
$$x^{p-1}a_1, \;\; ya_1 - xa_2, \;\; y^{p-1}a_2 - x^{p-1}a_3, \;\; ya_3 - xa_4, \;\; \dots, \;\; ya_{n-1} - xa_n, \;\; y^{p-1}a_n$$

for n even, and

$$xa_1, \;\; ya_1 - x^{p-1}a_2, \;\; y^{p-1}a_2 - xa_3, \;\; ya_3 - x^{p-1}a_4, \;\; \dots, \;\; y^{p-1}a_{n-1} - xa_n, \;\; ya_n$$

for n odd.

Let n be even. For illustrational purposes, we include a picture of $\Omega^4 k$ for $p = 3$,



The kernel of $\Theta_E = sx + ty$ on $k[s,t] \otimes \Omega^n k$ is a submodule of a free $k[s,t]$-module generated by the "middle layer" of $\Omega^n k$, that is, by $v_1 = x^{p-1}a_1, v_2 = x^{p-2}(ya_1 - xa_2), v_3 = x^{p-3}y(ya_1 - xa_2), \ldots, v_p = y^{p-2}(ya_1 - xa_2), v_{p+1} = y^{p-1}a_2 - x^{p-1}a_3, \ldots, v_{\frac{np}{2}+1} = y^{p-1}a_n$. Everything below the middle layer which is in $\operatorname{Ker}\Theta_E$ also lies in $\operatorname{Im}\Theta_E^{p-1}$. One verifies that the quotient

$$\operatorname{Ker}\{\Theta_E : k[s,t] \otimes \Omega^n k \to k[s,t] \otimes \Omega^n k\} / \operatorname{Im}\{\Theta_E^{p-1} : k[s,t] \otimes \Omega^n k \to k[s,t] \otimes \Omega^n k\}$$

is generated by

$$s^{\frac{np}{2}} v_1 - s^{\frac{np}{2}-1} t v_2 + \ldots \pm t^{\frac{np}{2}} v_{\frac{np}{2}+1}.$$

Arguing as in the proof of Prop. 6.13, we conclude that the corresponding locally free sheaf of rank 1 is $\mathcal{O}_{P^1}(-\frac{np}{2})$. The calculation for odd positive n is similar. For negative values of n, one verifies the formula by doing again a similar calculation with dual modules.

Now let $r > 2$, let $i : F \subset E$ be a subgroup scheme isomorphic to $\mathbb{G}_{a(1)}^{\times 2}$, and let $f : \mathbb{P}(E) \to \mathbb{P}(F)$ be the map induced by the embedding i. Since $(\Omega_E^n k)\downarrow_F \simeq \Omega_F^n k \oplus \text{proj}$, we conclude that $\mathcal{H}^{[1]}((\Omega_E^n k)\downarrow_F) \simeq \mathcal{H}^{[1]}(\Omega_F^n k)$. Proposition 5.3 implies an isomorphism $f^*(\mathcal{H}^{[1]}(\Omega_E^n k)) \simeq \mathcal{H}^{[1]}((\Omega_E^n k)\downarrow_F)$. Hence,

$$f^*(\mathcal{H}^{[1]}(\Omega_E^n k)) \simeq \mathcal{H}^{[1]}(\Omega_F^n k).$$

The proposition now follows from the observation that

$$f : \mathbb{P}(F) \simeq \mathbb{P}^1 \longrightarrow \mathbb{P}(E) \simeq \mathbb{P}^{r-1}$$

induces an isomorphism on Picard groups via f^*. $\square$

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