Π-SUPPORTS FOR MODULES FOR FINITE GROUP SCHEMES

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Abstract. We introduce the space $\Pi(G)$ of equivalence classes of π-points of a finite group scheme G, and associate a subspace $\Pi(G)_M$ to any G-module M. Our results extend to arbitrary finite group schemes G over arbitrary fields k of positive characteristic and to arbitrarily large G-modules the basic results about "cohomological support varieties" and their interpretation in terms of representation theory. In particular, we prove that the projectivity of any (possibly infinite dimensional) G-module can be detected by its restriction along π-points of G. Unlike the cohomological support variety of a G-module M, the invariant $M \mapsto \Pi(G)_M$ satisfies good properties for all modules, thereby enabling us to determine the thick, tensor-ideal subcategories of the stable module category of finite dimensional G-modules. Finally, using the stable module category of G, we provide $\Pi(G)$ with the structure of a ringed space which we show to be isomorphic to the scheme $\operatorname{Proj} \mathrm{H}^\bullet(G, k)$.

0. Introduction

In [14], we considered flat maps $\alpha : k[t]/t^p \to kG$ factoring through an abelian subgroup scheme $C \subset G$ of a finite group scheme G over an algebraically closed field k. Such maps were called "abelian p-points". As pointed out to us by Rolf Farnsteiner (cf [15]), our theory requires us to restrict consideration to flat maps α which factor through a *unipotent* abelian subgroup scheme. We call these more restricted maps "p-points" of G; all of the results of [14] are valid if "abelian p-point" is replaced by p-point. In particular, for a finite group scheme G over an algebraically closed field k, [14] introduces a space $P(G)$ of equivalence classes of p-points, with the equivalence relation determined in terms of the behaviour of restrictions of finite dimensional kG-modules. Furthermore, to a finite dimensional kG-module M, we associated a closed subspace $P(G)_M$. These invariants are generalizations of Carlson's rank variety for an elementary abelian p-group E and the cohomological support variety for a finite dimensional kE-module M [8],[2].

The purpose of this paper is to pursue further our point of view, thereby extending earlier results to any finite group scheme G over an arbitrary field k of characteristic $p > 0$ and to an arbitrary kG-module M. We suggest that our construction of "generalized p-points" (which we call "π-points") is both more natural and more intrinsic than previous considerations which utilized a combination of cohomological and representation-theoretic invariants.

The innovation which permits us to consider finite group schemes over an arbitrary field and their infinite dimensional (rational) representations is the consideration of equivalence classes of flat maps $K[t]/t^p \to KG_K$ which factor through

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some unipotent abelian subgroup scheme C_K of G_K not necessarily defined over k, where K/k is some field extension. Our fundamental result is Theorem 3.6 which asserts that for an arbitrary finite group scheme over a field k there is a natural homeomorphism

$$\Psi_G : \Pi(G) \xrightarrow{\sim} \mathrm{Proj}(\mathrm{H}^\bullet(G,k))$$

relating the space $\Pi(G)$ of π-points of G to the projectivization of the affine scheme given by the cohomology algebra $\mathrm{H}^\bullet(G,k)$. In other words, consideration of flat maps $K[t]/t^p \to KG_K$ for field extensions K/k enables us to capture the information encoded in the prime ideal spectrum of $\mathrm{H}^\bullet(G,k)$ rather than simply that of the maximal ideal spectrum. Indeed, we verify in Theorem 4.2 a somewhat sharper result, in that we determine (up to a purely inseparable field extension of controlled p-th power degree) the minimal field of definition of such a π-point in terms of its image under Ψ_G.

The need to consider such field extensions K/k when one considers infinite dimensional kG-modules had been recognized earlier. Nevertheless, our results improve upon results found in the literature for infinite dimensional modules for various types of finite group schemes over an algebraically closed field [3], [5], [6], [18], [22], [23]. Perhaps the most important and difficult of these results is Theorem 5.3 which asserts that the projectivity of any (possibly infinite dimensional) module M for an arbitrary finite group scheme G can be detected "locally" in terms of the restrictions of M along the π-points of G. This was proved for finite groups in [6], for unipotent group schemes in [3] and for infinitesimal group schemes in [23]. This, together with the consideration of certain infinite dimensional modules introduced by Rickard in [26], provides us with the tools to analyze the tensor-ideal thick subcategories of the stable category of finite dimensional kG-modules.

Our consideration of the (projectivization) of the prime ideal spectrum rather than the maximal ideal spectrum of $\mathrm{H}^\bullet(G,k)$ enables us to associate a good invariant (the Π-supports, $\Pi(G)_M \subset \Pi(G)$, of the kG-module M) to an arbitrary kG-module. This invariant $\Pi(G)_M$ is defined in module-theoretic terms, essentially as the "subset of those π-points at which M is not projective." Although $\Pi(G)_M$ corresponds naturally to the cohomological support variety of M whenever M is finite dimensional, it does not have an evident cohomological interpretation for infinite dimensional kG-modules. The difference in behaviour of this invariant for finite dimensional and infinite dimensional kG-modules is evident in Corollary 6.7 which asserts that *every* subset of $\Pi(G)$ is of the form $\Pi(G)_M$ for some kG-module M. Our analysis is somewhat motivated by and fits with the point of view of Benson, Carlson, and Rickard [6].

We establish in Theorem 6.3 a bijection between the tensor-ideal thick subcategories of the triangulated category $\mathrm{stmod}\,(G)$ of finite dimensional G-modules and subsets of $\Pi(G)$ closed under specialization. This theorem verifies the main conjecture of [18] (for ungraded Hopf algebras), a conjecture first formulated in [20] in the context of "axiomatic stable homotopy theory" and then considered in [18], [19]. As a corollary, we show that the lattice of thick, tensor-closed subcategories of the stable module category $\mathrm{stmod}\,(G)$ is isomorphic to the the lattice of thick, tensor-closed subcategories of $D^{perf}(\mathrm{Proj}\,\mathrm{H}^\bullet(G,k))$, the full subcategory of the derived category of coherent sheaves on $\mathrm{Proj}\,\mathrm{H}^\bullet(G,k)$ consisting of perfect complexes.

Finally, Theorem 7.1 demonstrates how the scheme structure of $\mathrm{Proj}\,\mathrm{H}^\bullet(G,k)$ can be realized using $\Pi(G)$ and the category $\mathrm{stmod}\,(G)$.

We remark that the consideration of π-points suggests the formulation of finer invariants than $\Pi(G)_M$ which would provide more information about a kG-module M. In a forthcoming paper [16], the authors and Andrei Suslin formulate the maximal Jordan type of a finite dimensional representation of a finite group scheme based on the point of view and results of this paper. This in turn enables the formulation of the non-maximal support variety of a G-module M which provides information complementary to that provided by $\Pi(G)_M$.

Throughout this paper, p will be a prime number and all fields considered will be of characteristic p. We shall typically denote by k an arbitrary field of characteristic p and denote by $\overline{k}$ an algebraic closure of k.

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1. Recollection of cohomological support varieties

Let G be a finite group scheme defined over a field k. Thus, G has a commutative coordinate algebra $k[G]$ which is finite dimensional over k and which has a coproduct induced by the group multiplication on G, providing $k[G]$ with the structure of a Hopf algebra over k. We denote by kG the k-linear dual of $k[G]$ and refer to kG as the group algebra of G. Thus, kG is a finite dimensional, co-commutative Hopf algebra over k.

Examples to keep in mind are that of a finite group π (so that $k\pi$ is the usual group algebra of π) and that of a finite dimensional, p-restricted Lie algebra g (so that the group algebra in this case can be identified with the restricted enveloping algebra of g). These are extreme cases: π is totally discrete (a finite, etale group scheme) and the group scheme $G_{(1)}$ associated to the (p-restricted) Lie algebra of an algebraic group over k is connected.

By definition, a G-module is a comodule for $k[G]$ (with its coproduct structure) or equivalently a module for kG. If M is a kG-module, then we shall frequently consider the cohomology of G with coefficients in M,

$$\mathrm{H}^*(G, M) \;\equiv\; \mathrm{Ext}^*_G(k, M).$$

If $p = 2$, then $\mathrm{H}^*(G, k)$ is itself a commutative k-algebra. If $p > 2$, then the even dimensional cohomology $\mathrm{H}^\bullet(G, k)$ is a commutative k-algebra. We denote by

$$\mathrm{H}^\bullet(G, k) = \begin{cases} \mathrm{H}^*(G, k), & \text{if } p = 2, \\ \mathrm{H}^{ev}(G, k) & \text{if } p > 2. \end{cases}$$

As shown in [17], the commutative k-algebra $\mathrm{H}^\bullet(G, k)$ is finitely generated over k. Following Quillen [24], we consider the maximal ideal spectrum of $\mathrm{H}^\bullet(G, k)$,

$$|G| \;\equiv\; \mathrm{Specm}\ \mathrm{H}^\bullet(G, k).$$

Following the work of Carlson [8] and others, for any finite dimensional kG-module M we consider

$$|G|_M = \text{Specm } \mathrm{H}^\bullet(G,k)/\operatorname{ann}_{\mathrm{H}^\bullet(G,k)} \operatorname{Ext}^*_G(M,M),$$

where the action of $\mathrm{H}^\bullet(G,k)$ on $\operatorname{Ext}^*_G(M,M)$ is via a natural ring homomorphism $\mathrm{H}^\bullet(G,k) \to \operatorname{Ext}^*_G(M,M)$ (so that this annihilator can be viewed more simply as the annihilator of $\mathrm{id_M} \in \mathrm{Ext^0_G(M,M)}$).

In this paper, we shall be interested in prime ideals which are not necessarily maximal. Indeed, this is the fundamental difference between this paper and [14]. We shall not give a special name for Spec $\mathrm{H}^\bullet(G,k)$, the scheme of finite type over k whose points are the prime ideals of $\mathrm{H}^\bullet(G,k)$ or to the scheme Spec $\mathrm{H}^\bullet(G,k)/\operatorname{ann}_{\mathrm{H}^\bullet(G,k)} \operatorname{Ext}^*_G(M,M)$, refinements of $|G|$ and $|G|_M$ respectively.

We shall often change the base field k via a field extension K/k. We shall use the notations

$$G_K = G \times_{\operatorname{Spec} k} \operatorname{Spec} K, \quad M_K = M \otimes_k K$$

to indicate the base change of the group scheme G over k and the base change of the kG-module M to a KG_K-module (where $KG_K = kG \otimes_k K$ will often be denoted KG).

In [28, 29], a map of schemes

$$\Psi_G : V_r(G) \to \text{Spec } \mathrm{H}^\bullet(G,k)$$

is exhibited for a finite, connected group scheme G over k and shown to be a homeomorphism. Here, $V_r(G)$ is the scheme of 1-parameter subgroups of G, a scheme representing a functor which makes no reference to cohomology. Moreover, this homeomorphism restricts to homeomorphisms

$$\Psi_G : V_r(G)_M \to \text{Spec } \mathrm{H}^\bullet(G,k)/\operatorname{ann}_{\mathrm{H}^\bullet(G,k)} \operatorname{Ext}^*_G(M,M)$$

for any finite dimensional kG-module M, where once again $V_r(G)_M$ is defined without reference to cohomology. One of the primary objectives of this paper is to extend this correspondence to all finite group schemes. Even for finite groups other than elementary abelian p-groups, such an extension has not been exhibited before.

2. π-points of G

We let G be a finite group scheme over a field k. In this section, we introduce our construction of the π-points of G and establish some of their basic properties. If $f : V \to W$ is a map of varieties or modules over k and if K/k is a field extension then we denote by $f_K = f \otimes 1_K : V_K \to W_K$ the evident base change of f. Given a map $\alpha : A \to B$ of algebras and a B-module M, we denote by $\alpha^*(M)$ the pull-back of M via α.

Our definition of π-point is an extension of our earlier definition of p-point (as corrected in [15]), now allowing extensions of the base field k. This enables us to consider finite group schemes defined over a field k which is not algebraically closed. Moreover, even if the base field k is algebraically closed, it is typically necessary to consider more "generic" maps $K[t]/t^p \to KG$ than those defined over k when considering infinite dimensional kG-modules.

We remind the reader that the representation theory of $K[t]/t^p$ is particularly simple: a $K[t]/t^p$-module is projective if and only if it is free; there are only finitely many indecomposable modules, one of dimension i for each i with $1 \leq i \leq p$.

Definition 2.1. Let G be a finite group scheme over k. A π-point of G (defined over a field extension K/k) is a (left) flat map of K-algebras

$$\alpha_K : K[t]/t^p \to KG$$

(i.e., a K-linear ring homomorphism with respect to which KG is flat as a left $K[t]/t^p$-module) which factors through the group algebra $KC_K \subset KG_K = KG$ of some unipotent abelian subgroup scheme C_K of G_K (with $C_K \to G_K$ defined over K, but not necessarily defined over k).

If $\beta_L : L[t]/t^p \to LG$ is another π-point of G, then α_K is said to be a *specialization* of β_L , written $\beta_L \downarrow \alpha_K$, provided that for any finite dimensional kG-module M, $\alpha_K^*(M_K)$ being free implies that $\beta_L^*(M_L)$ is free.

Two π-points $\alpha_K : K[t]/t^p \to KG$, $\beta_L : L[t]/t^p \to LG$ are said to be *equivalent*, written $\alpha_K \sim \beta_L$, if $\alpha_K \downarrow \beta_L$ and $\beta_L \downarrow \alpha_K$.

Observe that the condition that a π-point $\alpha_K : K[t]/t^p \to KG$ factors through the group algebra of a unipotent abelian subgroup scheme $C_K \subset G_K$ is the only aspect of the definition of a π-point which uses the Hopf algebra structure of kG. We point out that the homeomorphism of Theorem 3.6 requires consideration of π-points α_K which factor through the group algebra of unipotent abelian subgroup schemes $C_K \subset G_K$ defined over field extensions K/k of positive transcendence degree even in the case in which $G = SL_{2(1)}$ (the first infinitesimal subgroup scheme of the algebraic group SL_2, with group algebra the restricted enveloping algebra of sl_2).

In the following remark we demonstrate that the notion of specialization of π-points often has a familiar geometric interpretation.

Remark 2.2. Let R be a commutative Noetherian domain over k with a field of fractions K. Let $\alpha_R : R[t]/t^p \to RG$ be a flat map of R-algebras, and M be a kG-module of dimension m. Let $\alpha_K = \alpha_R \otimes_R K : K[t]/t^p \to KG$, and assume that α_K defines a π-point of G (i.e. we assume that α_K factors through a unipotent abelian subgroup scheme of G_K). The action of t on $\alpha_K^*(M_K)$ is given by some p-nilpotent matrix $A_\alpha \in M_m(R)$, and $\alpha_K^*(M_K)$ is free if and only if the Jordan form of the matrix A_α consists of Jordan blocks each of which are of size p if and only if the rank of A_α is $\frac{p-1}{p} \cdot m$.

Let $\phi : R \to \overline{k}$ be a map of k-algebras such that the base change of α_R via ϕ, $\alpha_\phi = \alpha_R \otimes_\phi \overline{k} : \overline{k}[t]/t^p \to \overline{k}G$, is a π-point of G. The action of t on $\alpha_\phi^*(M_{\overline{k}})$ is given by $(A_\alpha)_\phi = A_\alpha \otimes_\phi \overline{k} \in M_m(\overline{k})$, and, hence, $\alpha_\phi^*(M_{\overline{k}})$ is free as $\overline{k}[t]/t^p$-module if and only if the rank of $(A_\alpha)_\phi$ is $\frac{p-1}{p} \cdot m$. This is the case only if the rank of A_α is $\frac{p-1}{p} \cdot m$. Therefore, $\alpha_\phi^*(M_{\overline{k}})$ being free implies $\alpha_K^*(M_K)$ being free. Since this works for any module M, we conclude that α_ϕ is a specialization of α_K in the sense of Definition 2.1.

The following three examples involve sufficiently small finite group schemes G that their analysis is quite explicit. Nonetheless, the justification of the "genericity" assertions in these examples requires Theorem 3.6.

Example 2.3. Let G be the finite group $\mathbb{Z}/p \times \mathbb{Z}/p$, so that $kG \simeq k[x,y]/(x^p, y^p)$. A map $\alpha_K : K[t]/t^p \to KG$ is flat if and only if t is sent to a polynomial in x, y with non-vanishing linear term [14, 2.2]. Such a flat map α_K is equivalent to a flat map $\beta_K : K[t]/t^p \to KG$ if and only if $\alpha_K(t)$ and $\beta_K(t)$ have linear terms which are scalar multiples of each other [14, 2.2,2.6].

For example, a group homomorphism $\mathbb{Z}/p \to \mathbb{Z}/p \times \mathbb{Z}/p$ sending a generator σ of $\mathbb{Z}/p$ to (ζ^i, ξ^j) where ζ, ξ are generators of $\mathbb{Z}/p$, induces a map of group algebras

$$k[\sigma]/(\sigma^p - 1) \to k[\zeta, \xi]/(\zeta^p - 1, \xi^p - 1); \quad \sigma \mapsto \zeta^i \xi^j.$$

Viewed as a map of algebras, this is equivalent to $\alpha : k[t]/t^p \to k[x,y]/(x^p, y^p)$ sending t to $ix + jy$ since the images of the nilpotent generator under the two maps differ by a polynomial in the generators of the augmentation ideal without linear term.

Thus, any equivalence class has a representative which is given by a linear polynomial in x and y, unique up to scalar multiple. Let $K_0 = k(z,w)$, the field of fractions of the polynomial ring $k[z,w]$. Let $\eta_{K_0} : K_0[t]/t^p \to K_0[x,y]/(x^p, y^p)$ be the map that sends t to $zx + wy$. Then any flat map $\alpha : k[t]/t^p \to k[x,y]/(x^p, y^p)$ defined by sending t to a linear polynomial on x and y is a "specialization" of η_{K_0} in the sense that we get α via specializing z, w to some elements of k. This is easily seen to imply that $\eta_{K_0} \downarrow \alpha$.

Indeed, we can be more efficient in defining a "generic" π-point for G, for we observe that any $\alpha : k[t]/t^p \to k[x,y]/(x^p, y^p)$ defined by sending t to a linear polynomial in x and y is a "specialization" of

$$\xi_{k(z)} : k(z)[t]/t^p \to k(z)[x,y]/(x^p, y^p), \quad t \mapsto zx + y.$$

Namely, the flat map

$$\phi_{a,b} : k[t]/t^p \to k[x,y]/(x^p, y^p), \quad t \mapsto ax + by$$

with $a, b \in k$ is a specialization of $\xi_{k(z)}$: if $b \neq 0$ (respectively, $a \neq 0$), then $\phi_{a,b}$ is equivalent to the specialization of $\xi_{k(z)}$ obtained by setting $z = \frac{a}{b}$ (resp., replacing $\xi_{k(z)}$ by the equivalent $\xi'_{k(z)} : k(z) \to k(z)[x,y]/(x^p, y^p), t \mapsto x + \frac{1}{z}y$ and setting $1/z = \frac{b}{a}$).

We give a direct proof of the fact that any π-point $\phi_{a,b}$ is a specialization of $\xi_{k(z)}$ in the sense of Definition 2.1 (which follows in much greater generality from Corollary 4.3, for example). We assume $b \neq 0$. Let M be a kE-module and suppose $\phi^*_{a,b}(M)$ is free. Write $\phi^*_{a,b}(M) = \oplus\, (k[t]/t^p)\, e_i$, where $\{e_i\}$ for a basis for $\phi^*_{a,b}(M)$ as a free $k[t]/t^p$-module. Since $(ax + by)^{p-1} e_i \neq 0$ in M, we conclude that $(zx + y)^{p-1} e_i \neq 0$ in $M \otimes k(z)$. Therefore, $M \otimes k(z) \simeq \oplus\, k(z)[t]/t^p\, e_i$ and thus is free. In fact, we shall be able to conclude that any π-point α_K is a specialization of $\xi_{k(z)}$ in the sense of Definition 2.1.

Example 2.4. Let $E \cong (\mathbb{Z}/2)^{\times 3}$, char $k = 2$, $\{g_1, g_2, g_3\}$ be chosen generators of E. As in Example 2.3, any π-point of kE is a specialization of

$$\eta_{k(x,y)} : k(x,y)[t]/t^p \to k(x,y)E, \quad t \mapsto x(g_1 - 1) + y(g_2 - 1) + (g_3 - 1).$$

Let $M_{a,b,c}$ be a 4-dimensional kE-module indexed by the triple $a, b, c \in k$ with action of g_1, g_2, g_3 given by

$$g_1 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad g_1 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ 0 & b & 0 & 1 \end{bmatrix} \qquad g_1 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

The computation of [4, II.5.8] together with the homeomorphism of Theorem 3.6 implies that

$$\alpha_{u,v,w} : k[t]/t^p \to kE, \quad t \mapsto u(g_1 - 1) + v(g_1 - 1) + w(g_3 - 1)$$

satisfies the condition that $\alpha^*_{s,t,u}(M_{a,b,c})$ is not projective if and only if $\langle u, v, w\rangle \in \mathbb{P}^2$ lies on the quadric $Q_{a,b,c}$ defined as the locus of the homogeneous polynomial $(x + ay)(x + by) = cz^2$. (In the terminology to be introduced in Definition 3.1, $\Pi(E)_{M_{a,b,c}} \subset \Pi(E)$ equals the quadric $Q_{a,b,c}$).

Thus, for $c \neq 0$, every π-point of kE for which the restriction of $M_{a,b,c}$ is not projective is a specialization of the π-point given as

$$\alpha_{K_0} : K_0[t]/t^p \to K_0E; \qquad \alpha_{K_0}(t) = x(g_1 - 1) + y(g_2 - 1) + (g_3 - 1),$$

where $K_0 = frac\{k[x, y]/(x + ay)(x + by) - c\}$.

Example 2.5. Consider $G = (SL_2)_{(1)}$, the first infinitesimal kernel of the algebraic group SL_2, and assume that $p > 2$ for simplicity. Then the group algebra kG can be identified with the restricted enveloping algebra of sl_2, the (p-restricted) Lie algebra of 2×2 matrices of trace 0. We can explicitly describe kG as the (non-commutative) algebra given by

$$kG = k\{e, f, h\}/\langle e^p, f^p, h^p - h, he - eh - 2e, hf - fh + 2f, ef - fe - h\rangle.$$

Let K/k be a field extension. A choice of values $(E, F, H) \in K$, not all 0 and satisfying $H^2 = -EF$, determines a flat map

$$K[t]/t^p \to KG, \quad t \mapsto Ee + Ff + Hh.$$

If we let $x_{i,j}$ denote the natural coordinate functions on 2×2 matrices, then the variety of (p-) nilpotent elments is given by

$$N = \operatorname{Spec} k[x_{1,1}, x_{1,2}, x_{2,1}]/x^2_{1,1} + x_{1,2}x_{2,1}.$$

Let K_0 denote the field of fraction of N,

$$K_0 = frac\{k[x_{1,1}, x_{1,2}, x_{2,1}]/x^2_{1,1} + x_{1,2}x_{2,1}\}.$$

Then any flat map $K[t]/t^p \to KG$ is a specialization of the following "generic" flat map:

$$K_0[t]/t^p \to K_0G, \quad t \mapsto x_{1,2}e + x_{2,1}f + x_{1,1}h.$$

As in Example 2.3, we readily verify that we can more efficiently define this flat map as

$$frac\{k[x, y]/(1 + xy)\}[t]/t^p \to frac\{k[x, y]/(1 + xy)\}G, \quad t \mapsto ye + xf + h.$$

The proof of the following proposition follows immediately from the equality

$$(\alpha_\Omega)^*(M_\Omega) \;=\; (\alpha^*_K(M_K))_\Omega$$

for any triple $\Omega/K/k$ of field extensions and kG-module M and any π-point $\alpha_K : K[t]/t^p \to KG$.

Proposition 2.6. *Let G be a finite group scheme over a field k. Let $\alpha_K : K[t]/t^p \to KG$, $\beta_L : L[t]/t^p \to LG$ be π-points of G. Then the following conditions are equivalent:*

1. $\alpha_K \sim \beta_L$.
2. *For some field extension Ω/k containing both K and L, $\alpha_\Omega \sim \beta_\Omega$.*
3. *For any field extension Ω/k containing both K and L, $\alpha_\Omega \sim \beta_\Omega$.*

It is worth observing that the equivalence of $\alpha_\Omega, \beta_\Omega$ as π-points of G does not imply their equivalence as π-points of G_Ω (because for the latter one must test projectivity on all finite dimensional ΩG_Ω-modules and not simply those which arise from kG-modules). As we shall see, this can be reformulated as the observation that the space of π-points of G_Ω does not map injectively to the space of π-points of G. We discuss this further prior to Theorem 4.6.

The preceding proposition admits the following two corollaries concerning the naturality properties of π-points. The first follows immediately from the observation that the image under a map of group schemes of a unipotent abelian finite group scheme is once again a unipotent abelian finite group scheme. Namely, if $C' \to C$ is a quotient map of affine group schemes with C' a unipotent abelian finite group scheme, then $kC' \to kC$ is a surjective homomorphism (cf. [31, 15.1]); since kC' is commutative and local, so is kC. The second corollary is essentially a tautology, based on the observation that for field extensions $\Omega/L/K/k$, a π-point $\alpha_\Omega : \Omega[t]/t^p \to \Omega G$ of the group scheme G_L can be naturally viewed as a π-point of G_K.

Corollary 2.7. *Let $j : H \to G$ be a flat homomorphism of finite group schemes over a field k (i.e., assume with respect to the induced map $kH \to kG$ of group algebras that kG is flat as a left kH-module). Let $j_* : kH \to kG$ be the induced map on group algebras. The composition with j_* sending a π-point $\alpha_K : K[t]/t^p \to KH$ to $j_* \circ \alpha_K : K[t]/t^p \to KG$ induces a well defined map from the set of equivalence classes of π-points of H to the set of equivalence classes of π-points of G.*

Corollary 2.8. *Let G be a finite group scheme over the field k, $L/K/k$ be field extensions. Then the natural inclusion of the set of π-points $\alpha_\Omega : \Omega[t]/t^p \to \Omega G$ into the set of π-points $\beta_F : F[t]/t^p \to FG$, where Ω/L and F/K are field extensions, induces a well defined map from the set of equivalence classes of π-points of G_L to the set of equivalence classes of π-points of G_K.*

The following construction of a finite dimensional kG-module L_ζ associated to a (homogeneous) element $\zeta \in \mathrm{H}^\bullet(G,k)$ is due to J. Carlson [9]. We remind the reader of *Heller shifts* $\Omega^j(M)$ of a kG-module constructed in terms of a minimal projective resolution of M (cf. [4]). For $\zeta \in \mathrm{H}^{2i}(G,k)$, let L_ζ be the kG-module defined by the short exact sequence

$$0 \to L_\zeta \to \Omega^{2i}(k) \to k \to 0, \tag{2.8.1}$$

where the map $\Omega^{2i}(k) \to k$ represents $\zeta \in \mathrm{Hom}_G(\Omega^{2i}(k), k) = \mathrm{Ext}^{2i}_G(k,k)$.

These *Carlson modules* L_ζ will be used frequently in what follows.

Proposition 2.9. *Let G be a finite group scheme over a field k and let $\alpha_K : K[t]/t^p \to KG$ be a π-point of G. Let $\zeta \in \mathrm{H}^{2i}(G,k)$ and let* $\ker\{\alpha_K^*\}$ *denote the kernel of the algebra homomorphism $\alpha_K^* : \mathrm{H}^\bullet(G_K, K) \to \mathrm{H}^\bullet(K[t]/t^p, K)$.*

Then $\zeta \in \ker\{\alpha_K^*\} \cap \mathrm{H}^\bullet(G,k)$ if and only if $\alpha_K^*(L_{\zeta,K})$ is not projective as a $K[t]/t^p$-module, where we use $L_{\zeta,K}$ to denote $(L_\zeta)_K$.

Proof. Since the Heller operators commute with field extensions, $L_{\zeta_K} = L_{\zeta,K}$ as KG-modules, where for clarity we have used $\zeta_K \in \mathrm{H}^\bullet(G,K)$ to denote the image of $\zeta \in \mathrm{H}^\bullet(G,k)$. We apply the flat map α_K to the short exact sequence of KG-modules to obtain a short exact sequence of $K[t]/t^p$-modules:

$$0 \to \alpha_K^*(L_{\zeta,K}) \to \alpha_K^*(\Omega^{2i}(K)) \to K \to 0.$$

As argued in [14, 2.3], $\alpha_K^*(\zeta_K) \neq 0$ if and only if $\alpha_K^*(L_{\zeta,K}) = \alpha_K^*(L_{\zeta_K})$ is projective. □

We now present our cohomological reformulation of specialization of π-points of G.

Theorem 2.10. *Let G be a finite group scheme over k and α_K, β_L be two π-points of G. Then $\beta_L \downarrow \alpha_K$ if and only if*

$$(\ker\{\beta_L^*\}) \cap \mathrm{H}^\bullet(G,k) \ \subset \ (\ker\{\alpha_K^*\}) \cap \mathrm{H}^\bullet(G,k). \tag{2.10.1}$$

Proof. We first show the "only if" part. Let α_K be a specialization of β_L. Let ζ be any homogeneous element in $(\ker\{\beta_L^*\}) \cap \mathrm{H}^\bullet(G,k)$. By Proposition 2.9, $\beta_L^*(L_{\zeta,L})$ is not projective. Since $\beta_L \downarrow \alpha_K$, we conclude that $\alpha_K^*(L_{\zeta,K})$ is not projective. Applying 2.9 again, we get that $\zeta \in (\ker\{\alpha_L^*\}) \cap \mathrm{H}^\bullet(G,k)$. Since the ideals under consideration are homogeneous, the asserted inclusion follows.

Conversely, suppose α_K is not a specialization of β_L. By Proposition 2.6, we can assume that both α_K and β_L are defined over the same algebraically closed field Ω/k. Clearly, if we enlarge the field, the intersections $(\ker\{\beta_L^*\}) \cap \mathrm{H}^\bullet(G,k)$ and $(\ker\{\alpha_K^*\}) \cap \mathrm{H}^\bullet(G,k)$ do not change, so that we may assume that $K = L = \Omega$, with Ω algebraically closed.

Then, by Definition 2.1, there exists a finite dimensional kG-module M such that $\alpha_\Omega^*(M_\Omega)$ is projective but $\beta_\Omega^*(M_\Omega)$ is not. For a finite dimensional module, there is a natural isomorphism $\mathrm{Ext}^*_{G_\Omega}(M_\Omega, M_\Omega) \simeq \mathrm{Ext}^*_G(M,M)) \otimes_k \Omega$. Furthermore, since tensoring with Ω is exact, we have $\mathrm{ann}_{\mathrm{H}^\bullet(G,k)}(\mathrm{Ext}^*_G(M,M) \otimes_k \Omega) = \mathrm{ann}_{\mathrm{H}^\bullet(G_\Omega,\Omega)}(\mathrm{Ext}^*_{G_\Omega}(M_\Omega, M_\Omega))$.

Theorem [14, 4.11] now implies that

$$\mathrm{ann}_{\mathrm{H}^\bullet(G,k)}(\mathrm{Ext}^*_G(M,M)) \otimes_k \Omega = \mathrm{ann}_{\mathrm{H}^\bullet(G_\Omega,\Omega)}(\mathrm{Ext}^*_{G_\Omega}(M_\Omega, M_\Omega)) \subset \ker\{\beta_\Omega^*\}, \tag{2.10.2}$$

and

$$\mathrm{ann}_{\mathrm{H}^\bullet(G,k)}(\mathrm{Ext}^*_G(M,M)) \otimes_k \Omega = \mathrm{ann}_{\mathrm{H}^\bullet(G_\Omega,\Omega)}(\mathrm{Ext}^*_{G_\Omega}(M_\Omega, M_\Omega)) \not\subset \ker\{\alpha_\Omega^*\}. \tag{2.10.3}$$

Intersecting (2.10.2) with $\mathrm{H}^*(G,k)$, we get

$$\mathrm{ann}_{\mathrm{H}^\bullet(G,k)}(\mathrm{Ext}^*_G(M,M)) \subset \ker\{\beta_\Omega^*\} \cap \mathrm{H}^\bullet(G,k) \tag{2.10.4}$$

On the other hand, (2.10.3) implies that

$$\mathrm{ann}_{\mathrm{H}^\bullet(G,k)}(\mathrm{Ext}^*_G(M,M)) \not\subset \ker\{\alpha_\Omega^*\} \cap \mathrm{H}^\bullet(G,k). \tag{2.10.5}$$

Indeed, if this inclusion did hold, then by tensoring with Ω and then applying the fact that $(\ker\{\alpha_\Omega^*\} \cap \mathrm{H}^\bullet(G,k)) \otimes_k \Omega \subset \ker\{\alpha_\Omega^*\}$, we would get a contradiction to (2.10.3). Putting (2.10.4) and (2.10.5) together we get

$$(\ker\{\beta_\Omega^*\}) \cap \mathrm{H}^\bullet(G,k) \not\subset (\ker\{\alpha_\Omega^*\}) \cap \mathrm{H}^\bullet(G,k),$$

thereby proving the converse. □

As an immediate corollary, we add the following equivalent formulation of equivalence of π-points to those of Proposition 2.6 which will play a key role in the proof of our main theorem, Theorem 3.6.

Corollary 2.11. *Let G be a finite group scheme over k and α_K, β_L be two π-points of G. Then $\beta_L \sim \alpha_K$ if and only if*

$$(\ker\{\beta_L^*\}) \cap \mathrm{H}^\bullet(G,k) = (\ker\{\alpha_K^*\}) \cap \mathrm{H}^\bullet(G,k).$$

3. The homeomorphism $\Psi_G : \Pi(G) \to \operatorname{Proj} H^\bullet(G, k)$

In this section, we show for an arbitrary finite group scheme G over an arbitrary field k of characteristic $p > 0$ that the prime ideal spectrum of the cohomology ring can be described in terms of π-points of G. This is a refinement of [14] which provides a representation theoretic interpretation of the maximal ideal spectrum of the cohomology ring of G provided that k is algebraically closed.

The bijectivity of Theorem 3.6 below in the special case in which the finite group scheme is an elementary abelian p-group E and k is algebraically closed is equivalent to the foundational result of J. Carlson identifying the (maximal ideal) spectrum of $H^\bullet(E, k)$ with the rank variety of "shifted subgroups" of E [8]; the fact that this bijection is a homeomorphism in this special case is equivalent to "Carlson's Conjecture" proved by Avrunin and Scott [2]. In the special case in which G is connected, the homeomorphism of Theorem 3.6 is a weak form of the theorem of Suslin-Friedlander-Bendel which asserts that Spec $H^\bullet(G, k)$ is isogenous to the affine scheme of 1-parameter subgroups of G [28].

Let $\alpha_K : K[t]/t^p \to KG$ be a π-point, and denote by $\alpha_K^* : H^\bullet(G, K) \to H^\bullet(\mathbb{Z}/p, K)$ the induced map in cohomology. Let $\overline{K}$ be the algebraic closure of K. As it is shown in the proof of [14, 3.4], the map $\alpha_{\overline{K}}^*$ is finite and, hence, the kernel of this map, $\ker\{\alpha_{\overline{K}}^*\}$, is a homogeneous prime ideal strictly smaller than the augmentation ideal of $H^\bullet(G, \overline{K})$. Hence, $\ker\{\alpha_K^*\} = \ker\{\alpha_{\overline{K}}^*\} \cap H^\bullet(G, K)$ does not contain the augmentation ideal of $H^\bullet(G, K)$.

Definition 3.1. For any finite group scheme G over a field k, we denote by $\Pi(G)$ the set of equivalence classes of π-points of G,

$$\Pi(G) \equiv \{[\alpha_K];\ \alpha_K : K[t]/t^p \to KG \text{ is a } \pi\text{−point of } G\}.$$

For a finite dimensional kG-module M, we denote by

$$\Pi(G)_M \ \subset \ \Pi(G)$$

the subset of those equivalence classes $[\alpha_K]$ of π-points such that $\alpha_K^*(M_K)$ is not projective for any representative $\alpha_K : K[t]/t^p \to KG$ of the equivalence class $[\alpha_K]$. We say that $\Pi(G)_M$ is the Π-support of M.

Finally, we denote by

$$\Psi_G : \Pi(G) \to \operatorname{Proj} H^\bullet(G, k) \tag{3.1.1}$$

the *injective* map sending an equivalence class $[\alpha_K]$ of π-points to the homogeneous prime ideal $\ker\{\alpha_K^*\} \cap H^\bullet(G, k)$.

The fact that Ψ_G is well defined and injective is immediately implied by Theorem 2.10 and the above observation that $\ker\{\alpha_K^*\}$ is not the augmentation ideal of $H^\bullet(G, K)$ (so that $\ker\{\alpha_K^*\} \cap H^\bullet(G, k)$ is not the augmentation ideal of $H^\bullet(G, k)$).

Theorem 4.6 will enable us to retain in Definition 5.1 the same definition for kG-modules M which are possibly infinite dimensional. Moreover, Propositions 3.2 and 3.3 will remain valid for infinite dimensional kG-modules.

The following proposition, known as the "tensor product property", is somewhat subtle because a π-point $\alpha_K : K[t]/t^p \to KG$ need not respect the coproduct structure and thereby need not commute with tensor products. This tensor product property is one of the most important properties of Π-supports. The corresponding statement for cohomological support varieties has no known proof using only cohomological methods.

Proposition 3.2. *Let G be a finite group scheme over a field k and let M, N be finite dimensional kG-modules. Then*

$$\Pi(G)_{M\otimes N} = \Pi(G)_M \cap \Pi(G)_N.$$

Proof. For any π-point $\alpha : K[t]/t^p \to KG$ and any algebraically closed field extension Ω/k, $\alpha_K^*((M\otimes N)_K)$ is projective as a $K[t]/t^p$-module if and only if $\alpha_\Omega^*((M\otimes N)_\Omega)$ is projective as a $\Omega[t]/t^p$-module. On the other hand, [14, 3.9] asserts that $\alpha_\Omega^*((M\otimes N)_\Omega)$ is projective if and only if either $\alpha_\Omega^*(M_\Omega)$ or $\alpha_\Omega^*(N_\Omega)$ is projective which is the case if and only if either $\alpha_K^*(M_K)$ or $\alpha_K^*(N_K)$ is projective. □

We now provide a list of other properties of the association $M \mapsto \Pi(G)_M$ which follow naturally from our π-point of view. Namely, each of the properties can be checked one π-point at a time, thereby reducing the assertions to elementary properties of $K[t]/t^p$-modules.

Proposition 3.3. *Let G be a finite group scheme over a field k and let M_1, M_2, M_3 be finite dimensional kG-modules. Then*

1. $\Pi(G)_k = \Pi(G)$.
2. *If P is a projective kG-module, then $\Pi(G)_P = \emptyset$.*
3. *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact, then*
$$\Pi(G)_{M_i} \subset \Pi(G)_{M_j} \cup \Pi(G)_{M_k}$$
where $\{i,j,k\}$ is any permutation of $\{1,2,3\}$.
4. $\Pi(G)_{M_1\oplus M_2} = \Pi(G)_{M_1} \cup \Pi(G)_{M_2}$.

The topology we give to $\Pi(G)$ is the natural extension of that defined on the space $P(G)$ of p-points for G over an algebraically closed field given in [14, 3.10]. Observe that the formulation of this topology is given without reference to cohomology, although the verification that our topology satisfies the defining axioms of a topology does involve cohomology.

Proposition 3.4. *Let G be a finite group scheme over a field k. The class of subsets of $\Pi(G)$,*

$$\{\Pi(G)_M \subset \Pi(G) : M \text{ finite dimensional } G\text{−module}\},$$

is the class of closed subsets of a (Noetherian) topology on $\Pi(G)$.

Moreover, we have the equality

$$\Pi(G)_M = \Psi_G^{-1}(\operatorname{Proj}(\mathrm{H}^\bullet(G,k)/\operatorname{ann}_{\mathrm{H}^\bullet(G,k)}\operatorname{Ext}_G^*(M,M)))$$

for any finite dimensional kG-module M, where Ψ_G is the map of 3.1.1.

Proof. By Propositions 3.2 and 3.3, our class contains $\emptyset$, $\Pi(G)$ itself, and is closed under finite intersections and finite unions.

Observe that $\operatorname{Proj}\mathrm{H}^\bullet(G,k)$ is Noetherian and that each

$$\operatorname{Proj}(\mathrm{H}^\bullet(G,k)/\operatorname{ann}_{\mathrm{H}^\bullet(G,k)}\operatorname{Ext}_G^*(M,M)) \subset \operatorname{Proj}\mathrm{H}^\bullet(G,k)$$

is closed. Therefore, to complete the verification that we have given $\Pi(G)$ a Noetherian topology, it suffices to verify the asserted equality. This is equivalent to the following assertion for any finite dimensional kG-module M and any π-point $\alpha_K : K[t]/t^p \to KG$: namely, $\alpha_K^*(M_K)$ is not projective if and only if $\ker\{\alpha_K^*\}$ contains $\operatorname{ann}_{\mathrm{H}^\bullet(G,k)}\operatorname{Ext}_G^*(M,M)$. By base change from k to the algebraic closure of K, we may assume that k is algebraically closed and $K = k$. In this case, α_K is a p-point of G and the equality is verified (with $\Pi(G)_M \subset \Pi(G)$ replaced by $P(G)_M \subset P(G)$) in [14, 4.8] as corrected in [15]. □

Remark 3.5. We call $\Pi(G)$ with this topology the space of π-points of G.

We now verify that our space $\Pi(G)$ is related by a naturally defined homeomorphism to $\operatorname{Proj} \mathrm{H}^\bullet(G, k)$.

Theorem 3.6. *Let G be a finite group scheme over a field k, let $\operatorname{Proj} \mathrm{H}^\bullet(G, k)$ denote the space of homogeneous prime ideals (excluding the augmentation ideal) of the graded, commutative algebra $\mathrm{H}^\bullet(G, k)$ equipped with the Zariski topology, and let $\Pi(G)$ denote the set of π-points of G provided with the topology of Proposition 3.4.*

Then

$$\Psi_G : \Pi(G) \to \operatorname{Proj} \mathrm{H}^\bullet(G, k), \quad [\alpha_K] \mapsto \ker\{\alpha_K^*\} \cap \mathrm{H}^\bullet(G, k)$$

is a homeomorphism.

Moreover, if $j : H \to G$ is a flat homomorphism of finite group schemes over k, then the following square commutes:

(3.6.1)
$$\begin{array}{ccc} \Pi(H) & \xrightarrow{\Psi_H} & \operatorname{Proj} \mathrm{H}^\bullet(H, k) \\ \downarrow & & \downarrow \\ \Pi(G) & \xrightarrow{\Psi_G} & \operatorname{Proj} \mathrm{H}^\bullet(G, k) \end{array}$$

In this square, the left vertical arrow is given by Corollary 2.7 and the right vertical arrow by the map $\mathrm{H}^\bullet(H, k) \leftarrow \mathrm{H}^\bullet(G, k)$ induced by $H \to G$.

Furthermore, if K/k is a field extension, then the following square commutes:

(3.6.2)
$$\begin{array}{ccc} \Pi(G_K) & \xrightarrow{\Psi_{G_K}} & \operatorname{Proj} \mathrm{H}^\bullet(G_K, K) \\ \downarrow & & \downarrow \\ \Pi(G) & \xrightarrow{\Psi_G} & \operatorname{Proj} \mathrm{H}^\bullet(G, k) \end{array}$$

In this square, the left vertical arrow is given by Corollary 2.8 and the right vertical arrow by the base change map $\mathrm{H}^\bullet(G, k) \to \mathrm{H}^\bullet(G_K, K)$.

Proof. The verifications of the commutativity of squares (3.6.1) and (3.6.2) are straight-forward, and we omit them.

The injectivity of Ψ_G is given by Theorem 2.10 (as stated in Definition 3.1). To prove surjectivity, we consider a point $x \in \operatorname{Proj} \mathrm{H}^\bullet(G, k)$ with residue field $k(x)$ and base change to the algebraic closure K of $k(x)$, so that x is the image of a K-rational point $\overline{x} \in \operatorname{Proj} \mathrm{H}^\bullet(G_K, K)$. The commutativity of square (3.6.2) enables us to replace k by K, and thus reduces us to showing the surjectivity of Ψ_G on k-rational points, with k algebraically closed. This is proved in [14, 4.8].

The equality in the statement of Proposition 3.4 implies that the bijective map Ψ_G sends a closed subset (which by definition is of the form $\Pi(G)_M$) of $\Pi(G)$ to a closed subset of $\operatorname{Proj} \mathrm{H}^\bullet(G, k)$, thereby establishing the continuity of $(\Psi_G)^{-1}$.

To complete the proof that Ψ_G is a homeomorphism, it suffices to show that Ψ_G is continuous, i.e. that the preimage of any closed subset of $\operatorname{Proj} \mathrm{H}^\bullet(G, k)$ is closed in $\Pi(G)$. Hence, the theorem is implied by the following Proposition which is of interest on its own.

Proposition 3.7. *Let G be a finite group scheme over a field k and $I \subset \mathrm{H}^\bullet(G,k)$ be a homogeneous ideal generated by homogeneous elements $\zeta_1, \ldots, \zeta_n$. Then*

$$\Psi_G^{-1}(V(I)) \; = \; \Pi(G)_{L_{\zeta_1}\otimes\cdots\otimes L_{\zeta_n}}, \tag{3.7.1}$$

where $V(I) \subset \operatorname{Proj}\mathrm{H}^\bullet(G,k)$ is the zero locus of the homogeneous ideal I, and L_{ζ_i} are the Carlson modules as introduced in 2.8.1.

Proof. We first consider the case in which $I = \langle\zeta\rangle \subset \mathrm{H}^\bullet(G,k)$ is generated by a single element ζ. Let α_K be a π-point of G. The bijectivity of Ψ_G implies that $[\alpha_K] \in \Psi_G^{-1}(V(\langle\zeta\rangle)$ if and only if $\zeta \in \Psi_G([\alpha_K]) = \ker\{\alpha_K^*\} \cap \mathrm{H}^\bullet(G,k)$. By Proposition 2.9, $\zeta \in \ker\{\alpha_K^*\} \cap \mathrm{H}^\bullet(G,k)$ if and only if $\alpha_K^*(L_{\zeta,K})$ is not projective. We conclude that $[\alpha_K] \in \Psi_G^{-1}(V(\langle\zeta\rangle))$ if and only if $[\alpha_K] \in \Pi(G)_{L_\zeta}$. Hence, $\Psi_G^{-1}(V(\zeta)) \; = \Pi(G)_{L_\zeta}$.

Consequently, if I is generated by $\zeta_1, \ldots \zeta_n$, then we have

$$\Psi_G^{-1}(V(I)) = \Psi_G^{-1}(V(\langle\zeta_1\rangle) \cap \cdots \cap V(\langle\zeta_n\rangle)) = \Psi_G^{-1}(V(\langle\zeta_1\rangle)) \cap \cdots \cap \Psi_G^{-1}(V(\langle\zeta_n\rangle)) =$$
$$\Pi(G)_{L_{\zeta_1}} \cap \cdots \cap \Pi(G)_{L_{\zeta_n}} = \Pi(G)_{L_{\zeta_1}\otimes\cdots\otimes L_{\zeta_n}}$$

where the last equality is implied by the tensor product property (Proposition 3.2). □

Applying Ψ_G to the equality (3.7.1) and using Proposition 3.4 we get the following result which is an extension to prime ideal spectra of the corresponding result for k-rational points with k algebraically closed which is proved in [9] for finite groups and in [29] for infinitesimal group schemes.

Corollary 3.8. *Let G be a finite group scheme over a field k and $I \subset \mathrm{H}^\bullet(G,k)$ be a homogeneous ideal generated by homogeneous elements $\zeta_1, \ldots, \zeta_n$. Then*

$$V(I) = \operatorname{Proj}(\mathrm{H}^\bullet(G,k)/\operatorname{ann}_{\mathrm{H}^\bullet(G,k)} \operatorname{Ext}^*_G(M,M)))$$

where $V(I) \subset \operatorname{Proj}\mathrm{H}^\bullet(G,k)$ is the zero locus of the homogeneous ideal I, and $M = \otimes_i L_{\zeta_i}$.

4. Applications of the homeomorphism Ψ

In this section, we give some first applications of Theorem 3.6.

Remark 4.1. Let G be a finite group scheme over k and let A denote the coordinate algebra of G, $A = k[G]$. By definition, $\pi_0(G)$ is the spectrum of the maximal separable subalgebra of A. The projection $G \to \pi_0(G)$ admits a splitting if and only if the composition $G_{\mathrm{red}} \to G \to \pi_0(G)$ is an isomorphism; i.e., if and only if A modulo its nilradical $N \subset A$ is a separable algebra. The two conditions that the projection $G_F \to \pi_0(G_F)$ split and that $\pi_0(G_F)$ be constant are equivalent to the condition that A_F/N_F is isomorphic to a product of copies of F, where $A_F = A \otimes_k F$ and $N_F \subset A_F$ is the nilradical of A_F. Since $A_{\overline{k}}/N_{\overline{k}}$ is isomorphic to a product of copies of $\overline{k}$ (where $\overline{k}$ is an algebraic closure of k) and since A is finite dimensional over k, we may therefore choose some F/k finite over k such that the projection $G_F \to \pi_0(G_F)$ splits (so that G_F is a semi-direct product $G_F^0 \rtimes \pi_0(G_F)$) and that

$\pi_0(G_F)$ is a constant group scheme. By perhaps taking F to be a somewhat larger finite extension of k, we can insure that G_F^0 is geometrically connected (i.e., that the base change of G_F^0 to any extension L/F is connected).

Utilizing Theorem 3.6, we obtain the following result concerning the field of definition of a representative of a π-point α_K.

Theorem 4.2. *Let G be a finite group scheme over k and let F be a finite field extension F/k with the property that the projection $G_F \to \pi_0(G_F)$ splits and that $\pi_0(G_F)$ is a constant group scheme. Let r denote the height of the connected component $G^0 \subset G$.*

For any π-point $\alpha_K : K[t]/t^p \to KG_K$ of G, let $k_{[\alpha]}$ denote the residue field of $\Psi_G([\alpha_K]) \in \operatorname{Proj} \mathrm{H}^\bullet(G,k)$. Then α_K is equivalent to some π-point $\beta_L : L[t]/t^p \to LG_L$ with L a purely inseparable extension of degree $\leq p^r$ of the composite $F \cdot k_{[\alpha]}$.

Proof. To prove the proposition we may replace G by G_F; in other words, we may (and will) assume that $G \simeq G^0 \rtimes \pi_0(G)$ with $\pi_0(G)$ constant and G^0 geometrically connected. We consider some π-point $\alpha_K : K[t]/t^p \to KG_K$ of G.

Let τ denote the finite group $\pi_0(G)$. Suslin's detection theorem [27] asserts that modulo nilpotents any homogeneous element of $\mathrm{H}^\bullet(G,k)$ has a non-zero restriction via some group homomorphism of the form $\mathbb{G}_{a(r)L} \times E \to G_L$ for some field extension L/k and some elementary abelian subgroup $E \subset \tau$. Since $G = G^0 \rtimes \tau$ such a map must factor through some subgroup of G of the form $(G^0)^E \times E$. Consequently, the natural map

$$\mathrm{H}^\bullet(G,k) \to \bigoplus_{E \subset \tau} \mathrm{H}^\bullet((G^0)^E \times E, k)$$

has nilpotent kernel, where the sum is indexed by conjugacy classes of elementary abelian p-subgroups of τ. This implies that any point of $\operatorname{Proj} \mathrm{H}^\bullet(G,k)$ lies in the image of $\operatorname{Proj} \mathrm{H}^\bullet((G^0)^E \times E, k)$ for some elementary abelian p-subgroups $E \subset \tau$. The naturality of the homeomorphism Ψ_G of Theorem 3.6 (with respect to $(G^0)^E \times E \to G$) implies that $[\alpha_K]$ lies in the image of $\Pi((G^0)^E \times E)$ for such an elementary abelian p-group $E \subset \tau$.

Since the height of any infinitesimal subgroup scheme $(G^0)^E \subset G^0$ is at most r, it suffices to consider group schemes of the form $G' = (G')^0 \times E$ for some elementary abelian p-group E of rank s. Let r' be the height of the connected component $(G')^0$.

Assume first that $(G')^0$ is trivial, so that $G' = E$. Then a choice of generators for E determines the rank variety $V(E)$ and we can identify $\operatorname{Proj}(V(E))$ with $\Pi(E)$ – namely, each shifted cyclic subgroup of KE is a π-point of E, and we can represent any equivalence class of π-points by such a cyclic shifted subgroup. Then, the homeomorphism $\Psi_E : \Pi(E) \simeq \operatorname{Proj} \mathrm{H}^\bullet(E,k)$ refines to an isomorphism of k-algebras $k[x_1,\ldots,x_s] \cong \mathrm{H}^\bullet(E,k)_{red}$. Here, the coordinate algebra of the rank variety is identified with $k[x_1,\ldots,x_s]$, so that a shifted cyclic subgroup $\sum_{i=1}^s a_i(g_i - 1)$ is identified with $\sum_{i=1}^s a_i$, where $\{g_i,\ldots,g_s\}$ is a fixed choice of generators of E; the map $k[x_1,\ldots,x_s] \to \mathrm{H}^\bullet(G,k)_{red}$ is given by sending x_i to the dual of g_i if $p = 2$ and to the Bockstein of the dual of g_i if $p > 2$. In particular, any π-point $\alpha_K : K[t]/t^p \to KE$ can be represented by a π-point defined over $k_{[\alpha]}$.

Assume now that $s = 0$, so that $(G')^0 = G'$. Let $V((G')^0)$ denote the scheme of 1-parameter subgroups of $(G')^0$. By [29, 5.5], there is a natural k-algebra homomorphism

$$\psi : \mathrm{H}^\bullet((G')^0, k) \to k[V((G')^0)]$$

the image of which contains $k[V((G')^0)]^{p^r}$. Thus, the bijective map $\Psi_{(G')^0}$: $V((G')^0) \to \mathrm{Spec}\, \mathrm{H}^\bullet((G')^0, k)$ induces a map on residue fields which is an isomorphism up to a purely inseparable extension of degree at most p^r. This clearly implies the same assertion for $\Psi_{(G')^0} : \mathrm{Proj}\, V((G')^0) \to \mathrm{Proj}\, \mathrm{H}^\bullet((G')^0, k)$. We conclude that any π-point $\alpha_K : K[t]/t^p \to K(G')^0$ can be represented by a π-point defined over a purely inseparable extension of $k_{[\alpha]}$ of degree at most p^r.

More generally, consider $G' = (G')^0 \times E$. For any (scheme-theoretic) point $0 \neq x = (x_1, x_2) \in \mathrm{Spec}\,(\mathrm{H}^\bullet((G')^0, k) \otimes \mathrm{H}^\bullet(E, k))$, $k(x)$ equals the composite (inside some universal field extension of k) of $k(x_1)$ and $k(x_2)$, and moreover $k(x)$ is the residue field of the corresponding point of $\mathrm{Proj}\, \mathrm{H}^\bullet(G, k)$. As argued in [14, 4.1], every equivalence class of π-points of G is represented by a sum of π-points of the form $\beta_F \otimes 1 + 1 \otimes \gamma_L$ for π-points β_F, γ_L of $(G')^0, E$ respectively. Thus, this general case follows from the two special cases considered above.

□

Essentially by definition, the condition (2.10.1):

$$(\ker\{\beta_L^*\}) \cap \mathrm{H}^\bullet(G, k) \;\subset\; (\ker\{\alpha_K^*\}) \cap \mathrm{H}^\bullet(G, k),$$

holds if and only if $(\ker\{\alpha_K^*\}) \cap \mathrm{H}^\bullet(G, k)$ lies in the closure of $(\ker\{\beta_L^*\}) \cap \mathrm{H}^\bullet(G, k)$ as points of $\mathrm{Proj}\, \mathrm{H}^\bullet(G, k)$. Thus, Theorems 2.10 and 3.6 imply the following topological interpretation of specialization of π-points.

Proposition 4.3. *Let G be a finite group scheme over k, and let α_K, β_L be π-points of G. Then $\beta_L \downarrow \alpha_K$ if and only if $\Psi_G(\alpha_K) \in \mathrm{Proj}\, \mathrm{H}^\bullet(G, k)$ lies in the closure of $\Psi_G(\beta_L)$.*

Consequently, the set of π-points of G which are specializations of a given π-point α_K form a closed subset $\overline{\{[\alpha_K]\}} \subset \Pi(G)$.

Proposition 4.4. *Let k/k' be a field extension and $\sigma : k \to k$ a field automorphism over k'. Assume that the finite group scheme G over k is defined over k', so that $G = G' \times_{k'} \mathrm{Spec}\, k$ for some group scheme G' defined over k'. Then there is a natural action of σ on $\Pi(G)$, $[\alpha] \mapsto [\alpha^\sigma]$, which commutes with the homeomorphism $\Psi_G : \Pi(G) \to \mathrm{Proj}\, \mathrm{H}^\bullet(G, k)$, where the action on the right is induced by the map*

$$\sigma \otimes 1 : \mathrm{H}^\bullet(G, k) = k \otimes_{k'} \mathrm{H}^\bullet(G', k') \;\to\; k \otimes_{k'} \mathrm{H}^\bullet(G', k') = \mathrm{H}^\bullet(G, k).$$

Moreover, if M is a kG-module defined over k', and $\alpha_K : k[t]/t^p \to KG_K$ is a π-point, then $(\alpha_K^\sigma)^*(M_K)$ is projective if and only if $\alpha_K^*(M_K)$ is projective.

Proof. Let $\alpha_K : K[t]/t^p \to KG$ be a π-point of G. By replacing K/k by a finite extension of K if necessary, we may assume that the automorphism σ of k/k' extends to an automorphism $\tilde{\sigma} : K \to K$ over k'. Then $\tilde{\sigma}$ defines a map of k'-algebras

$$\tilde{\sigma} : KG = K \otimes_{k'} k'G' \xrightarrow{\tilde{\sigma}\otimes 1} K \otimes_{k'} k'G' = KG$$

We define $\alpha_K^{\tilde{\sigma}} : K[t]/t^p \to KG$ to be the K-algebra map which sends t to $(\alpha_K(t))^{\tilde{\sigma}} = \tilde{\sigma}(\alpha_K(t))$. Since $\tilde{\sigma} : KG \to KG$ induces a map in cohomology

$$\mathrm{H}^\bullet(G_K, K) = K \otimes_{k'} \mathrm{H}^\bullet(G', k') \xrightarrow{\tilde{\sigma}\otimes 1} K \otimes_{k'} \mathrm{H}^\bullet(G', k') = \mathrm{H}^\bullet(G_K, K),$$

which is again twisting by $\tilde{\sigma}$ we get

$$\ker\{(\alpha_K^{\tilde{\sigma}})^*\} = (\ker\{\alpha_K^*\})^{\tilde{\sigma}},$$

where we denote by $\mathcal{P}^{\tilde\sigma}$ the image of a homogeneous prime ideal $\mathcal{P} \subset \mathrm{H}^\bullet(G_K, K)$ under the action of $\tilde\sigma$. Since $\mathrm{H}^\bullet(G_K, K) \stackrel{\tilde\sigma\otimes 1}{\to} \mathrm{H}^\bullet(G_K, K)$ restricts to $\mathrm{H}^\bullet(G, k) \stackrel{\sigma\otimes 1}{\to} \mathrm{H}^\bullet(G, k)$, we further conclude that

$$\Psi_G([\alpha_K^{\tilde\sigma}]) = \ker\{(\alpha_K^{\tilde\sigma})^*\} \cap \mathrm{H}^\bullet(G, k)) = (\ker\{(\alpha_K)^*\})^{\tilde\sigma} \cap \mathrm{H}^\bullet(G, k)) =$$

$$(\ker\{\alpha_K^*\} \cap \mathrm{H}^\bullet(G, k))^\sigma = (\Psi_G([\alpha_K]))^\sigma$$

Since Ψ_G is an isomorphism on the equivalence classes of π-points, we get that sending α_K to $\alpha_K^{\tilde\sigma}$ determines a well defined action on $\Pi(G)$: $[\alpha_K] \mapsto [\alpha_K^{\tilde\sigma}]$. Moreover, the action does not depend upon the choice of extension $\tilde\sigma$ of σ, and is compatible with the homeomorphism Ψ_G.

Let M be a kG-module defined over k' and write $M = k \otimes_{k'} M'$. If $\rho : k'G' \to \mathrm{End}_{k'}(M')$ specifies the $k'G'$-module M', then $\rho_K(\alpha_K^{\tilde\sigma}(t))$ when viewed as a matrix is simply the result of applying $\tilde\sigma$ to the matrix entries of $\rho_K(\alpha_K(t))$. Consequently, we see that $(\alpha_K^{\tilde\sigma})^*(M_K) \cong (\alpha_K)^*(M_K)$. Thus, $\alpha_K^*(M_K)$ is free if and only if $(\alpha_K^{\tilde\sigma})^*(M_K)$ is free.

□

Let $p \in \mathrm{Proj}\,\mathrm{H}^\bullet(G, k)$ be a closed point which is rational over a finite separable extension F/k but is not k-rational, and let $\tilde p, \tilde q \in \mathrm{Proj}\,\mathrm{H}^\bullet(G_F, F)$ be distinct points mapping to p. Choose π-points $\alpha_K : K[t]/t^p \to KG$, $\beta_L : L[t]/t^p \to LG$ with the property that $\Psi_{G_F}([\alpha_K]) = \tilde p$, $\Psi_{G_F}([\beta_L]) = \tilde q$. Then for every finite dimensional kG-module M, $\alpha_K^*(M_K)$ is projective if and only if $\beta_L^*(M_L)$ is projective; however, there exists a finite dimensional FG_F-module N such that $\alpha_K^*(N_K)$ is projective and $\beta^*(N_L)$ is not projective.

To further illustrate the behaviour of the map $\Pi(G_K) \to \Pi(G)$ of Corollary 2.8, we determine the pre-images of this map in the special case of Example 2.3.

Example 4.5. We adopt the notation and conventions of Example 2.3 and let $K = k(z)$, the field of fractions of "generic" π-point of $G = \mathbb{Z}/p \times \mathbb{Z}/p$. As established in Example 2.3,

$$\xi_{k(z)} : k(z)[t]/t^p \to k(z)[x, y]/(x^p, y^p), \quad t \mapsto zx + y$$

represents the unique equivalence class of "generic" π-points of G. One readily observes that a π-point of G defined by $t \mapsto f(z)x + y$ with f any non-constant rational function f is equivalent to $\xi_{k(z)}$. However, points corresponding to distinct non-constant functions f are not equivalent as π-points of G_K (by [14, 2.2]). Thus the pre-image of the generic point of G under the map $\Pi(G_K) \to \Pi(G)$ has closed points in one-to-one correspondence with elements of $K^* - k^*$. On the other hand, a closed point of $\Pi(G)$ is represented by a flat map of the form

$$k[t]/t^p \to k[x, y]/(x^p, y^p), \quad t \mapsto ax + by$$

with at least one of $a, b \in k$ non-zero. The pre-image of such a point in $\Pi(G_K)$ consists of a single element, the equivalence class of

$$K[t]/t^p \to K[x, y]/(x^p, y^p), \quad t \mapsto ax + by.$$

More generally, the pre-image of $\Pi(G_K) \to \Pi(G)$ above some $[\alpha_K] \in \Pi(G)$ is non-empty, and any point of this pre-image has closure in $\Pi(G_K)$ with dimension at most the transcendence degree of the residue field of $[\alpha_K]$ over k. This last statement can be verified using the homeomorphism Ψ of Theorem 3.6.

In view of this observation of non-injectivity of the functorial map $\Pi(G_F) \to \Pi(G)$ for a field extension F/k, the following result is somewhat striking.

Theorem 4.6. *Let G be a finite group scheme over a field k. We say that two π-points $\alpha_K : K[t]/t^p \to KG$, $\beta_L : L[t]/t^p \to LG$ are strongly equivalent if for any (possibly infinite dimensional) kG-module M $\alpha_K^*(M_K)$ is projective if and only if $\beta_L^*(M_L)$ is projective.*

If $\alpha_K \sim \beta_L$, then α_K is strongly equivalent to β_L.

Proof. We first prove the statement in the special case when $L = K = k$, with k algebraically closed.

We quote here the statement of [14, 2.2] which will be used extensively throughout the proof: let M be a k vector space and α, β and γ be pair-wise commuting endomorphisms of M such that α, β are p-nilpotent and γ is p^r-nilpotent for some $r \geq 1$. Then M is free as a $k[u]/u^p$-module via the action of α if and only if M if free via the action of $\alpha + \beta\gamma$.

Let C be a unipotent abelian subgroup scheme of G. Thus, C is co-connected; i.e., the dual $C^\#$ (whose coordinate algebra is kC) is connected. The structure theorem for connected finite group schemes [31, 14.4] implies that $kC \simeq k[t_1, t_2, \dots t_n]/(t_1^{p^{i_1}}, \dots t_n^{p^{i_n}})$. By [14, 4.11], the space of equivalence classes of p-points of C is homeomorphic to $\operatorname{Proj} \mathrm{H}^\bullet(C, k)$, which in turn is homeomorphic to $\mathbb{P}_k^{n-1}$. Let $[\alpha_1 : \cdots : \alpha_n]$ be a point representing an equivalence class of p-points of C. Let $\alpha : k[t]/t^p \to kC$ be a p-point given by the formula

$$\alpha(t) = \alpha_1 t_1^{p^{i_1 - 1}} + \cdots + \alpha_n t_n^{p^{i_n - 1}}$$

and let $\beta : k[t]/t^p \to kC$ be an arbitrary representative of the same equivalence class.

As seen in [14], distinct linear terms of flat maps $k[t]/t^p \to kC$ give distinct maps in cohomology, polynomials without linear terms correspond to non-flat maps which are zero in cohomology, and the identification of non-zero linear terms corresponds to taking $\operatorname{Proj}(-)$. Hence, β is given by the formula

$$\beta(t) = c(\alpha_1 t_1^{p^{i_1 - 1}} + \dots, + \alpha_n t_n^{p^{i_n - 1}}) + p(t_1, t_2, \dots, t_n)$$

where c is a non-zero scalar, and $p(t_1, t_2, \dots, t_n)$ is a sum of monomials each one of which is a product of the term of the form $t_j^{p^{i_j - 1}}$ for some j and at least one other term of degree at least 1. Since Proposition [14, 2.2] quoted above applies to a possibly infinite dimensional k-vector space, this proposition implies that α is strongly equivalent to β.

We thereby conclude that equivalence implies strong equivalence for unipotent abelian finite group schemes. Applying [14, 4.2], we get that any p-point $\alpha : k[t]/t^p \to kC$ is equivalent and thus strongly equivalent to a p-point factoring through a quasi-elementary abelian subgroup scheme, i.e. a subgroup scheme isomorphic to $\mathbb{G}_{a(r)} \times E$ where E is an elementary abelian p-group.

By definition, any p-point of an arbitrary finite group scheme G over an algebraically closed field k factors through some unipotent abelian subgroup scheme of G. As argued above, any such p-point is strongly equivalent to one factoring through some quasi-elementary abelian subgroup scheme of G. Consider equivalent p-points of G , α and β, each of which factors through some quasi-elementary abelian subgroup scheme of G. Let G^0 be the connected component of G and $\pi = \pi_0(G)$

be the group of connected components. Corollary [14, 4.7] implies that α, β are conjugate by an element of π to equivalent p-points which factor through the same subgroup scheme $(G^0)^E \times E$, where $E \subset \pi$ is an elementary abelian subgroup of π. Since conjugation by elements of π does not change the strong equivalence class of a p-point, we are further reduced to the case in which G is of the special form $G' \times E$ with G' connected. Since $kE \simeq k\mathbb{G}_{a(r)}$, we may further assume that G itself is connected.

In this case, write α as the composition of some $\alpha_C : k[t]/t^p \to kC$ with C a connected unipotent abelian subgroup scheme of G and $kC \to kG$ induced by $\gamma : C \subset G$. By [14, 3.8], α_C is equivalent as a p-point of C to a composition of the form $\phi_* \circ \epsilon_r : k[t]/t^p \to k\mathbb{G}_{a(r)} \to kC$, where $\epsilon_r : k[t]/t^p \to k\mathbb{G}_{a(r)} \simeq k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p)$ is the algebra map sending t to u_{r-1} and ϕ_* is induced by a homomorphism of group schemes $\phi : \mathbb{G}_{a(r)} \to C$. Since equivalence implies strong equivalence for p-points of the unipotent abelian group scheme C, we conclude that α is strongly equivalent to $(\tilde{\alpha})_* \circ \epsilon_r$ where $\tilde{\alpha} = \gamma \circ \psi : \mathbb{G}_{a(r)} \to G$ is a one-parameter subgroup of G. Similarly, β is strongly equivalent to $(\tilde{\beta})_* \circ \epsilon_s$ where $\tilde{\beta} : \mathbb{G}_{a(s)} \to G$ is a one-parameter subgroup of G. By replacing $\tilde{\alpha}$ by a one-parameter subgroup obtained by precomposing $\tilde{\alpha}$ with the natural projection $\mathbb{G}_{a(r+s)} \to \mathbb{G}_{a(r)}$, and similarly for $\tilde{\beta}$, we may assume $r = s$. Yet for p-points of this special form to be equivalent they must be differ by scalar multiples by [14, 3.8] and thus are necessarily strongly equivalent.

Next, we show how to drop the condition that k be algebraically closed. Let Ω/k be an algebraically closed field of transcendence degree at least the Krull dimension of $\mathrm{H}^\bullet(G,k)$. In view of Proposition 2.6, the bijectivity of Ψ_G, and Theorem 4.3, we then may assume $L = K = \Omega$. Corollary 2.11 implies that $(\ker\{\beta_\Omega^*\}) \cap \mathrm{H}^\bullet(G,k) = (\ker\{\alpha_\Omega^*\}) \cap \mathrm{H}^\bullet(G,k)$. Let F denote the residue field of $\mathrm{H}^\bullet(G,k)$ at this prime ideal. Consider the compositions

$$\mathrm{H}^\bullet(G,k) \to \mathrm{H}^\bullet(G,\Omega) \rightrightarrows \mathrm{H}^\bullet(\Omega[t]/t^p, \Omega) \to \Omega$$

of the base change $\mathrm{H}^\bullet(G,k) \to \mathrm{H}^\bullet(G,k) \otimes_k \Omega \cong \mathrm{H}^\bullet(G,\Omega)$ with $\alpha_\Omega^*, \beta_\Omega^*$ and with evaluation at $T = 1$ of the polynomial algebra $\mathrm{H}^\bullet(\Omega[t]/t^p, \Omega) \cong \Omega[T]$. These compositions factor through F and determine two embeddings of F into Ω which are related by an element $\sigma \in Gal(\Omega/F)$. So defined, σ satisfies

$$\ker\{\beta_\Omega^*\} = (\ker\{\alpha_\Omega^*\})^\sigma.$$

Since $\Psi_{G_\Omega} : \Pi(G_\Omega) \to \mathrm{Proj}\,\mathrm{H}^\bullet(G_\Omega, \Omega)$ commutes with the action of σ by Proposition 4.4, we have the equality

$$(\ker\{\alpha_\Omega^*\})^\sigma = \ker\{(\alpha_\Omega^\sigma)^*\},$$

and thus $\alpha_\Omega^\sigma \sim \beta_\Omega$ as p-points of G_Ω.

Thus, the special case verified above in which $L = K = k$ is algebraically closed implies that for any ΩG-module N, $(\alpha_\Omega^\sigma)^* N$ is projective if and only if $(\beta_\Omega)^* N$ is projective. On the other hand, Proposition 4.4 implies that for a kG-module M, $(\alpha_\Omega^\sigma)^*(M_\Omega)$ is projective if and only if $\alpha_\Omega^*(M_\Omega)$ is projective. Hence, $\beta_\Omega^*(M_\Omega)$ is projective if and only if $\alpha_\Omega^*(M_\Omega)$ is projective for any kG-module M. In other words, α_K is strongly equivalent to β_L.

□

In the next proposition, we give several characterizations of closed points of $\Pi(G)$. In particular, if k is algebraically closed, then the space $P(G)$ of p-points is exactly the subspace of closed points of $\Pi(G)$.

Proposition 4.7. *Let G be a finite group scheme over a field k. Then the following conditions are equivalent on a π-point $\alpha_K : K[t]/t^p \to KG$ of G.*

1. *The equivalence class $[\alpha_K]$ of α_K is a closed point of $\Pi(G)$.*
2. *Any specialization of α_K is equivalent to α_K.*
3. *α_K is equivalent to some π-point $\beta_F : F[t]/t^p \to FG$ with F/k finite. In particular, if k is algebraically closed, then the equivalence class of α_K, $[\alpha_K]$ is represented by a map of the form $\beta : k[t]/t^p \to kG$.*
4. *There exists some finite dimensional non-projective kG-module M such that whenever $\beta_L : L[t]/t^p \to LG$ is a π-point with $\beta_L^*(M_L)$ not projective then α_K is equivalent to β_L.*

Proof. Granted the topology on $\Pi(G)$ given in Proposition 3.4, a π-point α is a specialization of a π-point β if and only if α is in the closure of β. Thus, (1) and (2) are equivalent.

If $\alpha_K : K[t]/t^p \to KG$ is a π-point, then $\ker\{\alpha_K^*\} \cap H^\bullet(G,k) \in \operatorname{Proj} \mathrm{H}^\bullet(G,k)$ is defined over K. Consequently, (3) implies (1), for any point of $\operatorname{Proj} \mathrm{H}^\bullet(G,k)$ defined over an algebraic extension of k must be closed. Conversely, let $\overline{k}$ denote the algebraic closure of k. Using Theorem 3.6, we see that any closed point of $\Pi(G)$ lies in the image of a closed point of $\Pi(G_{\overline{k}})$ which corresponds (naturally and bijectively) to a rational point of $\operatorname{Proj} \mathrm{H}^\bullet(G_{\overline{k}}, \overline{k})$ which corresponds (naturally and bijectively) to a p-point of $G_{\overline{k}}$ by [14, 4.6]. Any such p-point $\alpha_{\overline{k}} : \overline{k}[t]/t^p \to \overline{k}G_{\overline{k}}$ is defined over some finite extension of k.

The existence of a module M with the property described in (4) implies that for any β_L such that $\beta_L^*(M)$ is not projective, we have $[\beta_L] = [\alpha_K]$. Hence, $\Pi(G)_M \subset \{[\alpha_K]\}$. Since M is not projective, we conclude that $\Pi(G)_M$ coincides with $\{[\alpha_K]\}$. Therefore, $[\alpha_K]$ is closed by the definition of the topology on $\Pi(G)$. Conversely, if a point $[\alpha_K]$ of $\Pi(G)$ is closed then there exists a finitely generated non-projective kG-module M with $\Pi(G)_M = \{[\alpha_K]\}$. It is immediate to check that such M satisfies the required property. Hence, (1) is equivalent to (4).

□

We shall give an enhanced version of the "Quillen decomposition" of $\Pi(G)$, thereby refining the corresponding decomposition given in [14, 5.3] (stated for p-points, with k algebraically closed) and implicitly clarifying the somewhat ambiguous statement [14, 4.7].

Let G be a finite group scheme of the form $G^0 \rtimes \tau$, where $G^0 \subset G$ is the connected component of G which we assume to be geometrically connected and $\tau = \pi_0(G)$ is the (discrete) group of connected components of G. Observe that our assumption implies that G^0_K is connected for any field extension K/k.

We shall make use of the following terminology.

Definition 4.8. Let $[\alpha_K]$ be an equivalence class of π-points of G. A representative α_K is called *minimal* if α_K factors through $(G^0)^E \times E$ for some elementary abelian subgroup $E \subset \tau$ but there is no representative of the same equivalence class which factors through $(G^0)^E \times E'$ for some E' a proper subgroup of E.

Remark 4.9. Proposition [14, 4.2] implies that any equivalence class admits a representative which factors through a subgroup scheme isomorphic to $\mathbb{G}_{a(r)} \times E$. Since E has only finitely many subgroups, we conclude that there is always a minimal representative for any equivalence class of π-points.

Conjugation by elements of $\pi_0(G)$ induces an action on π-points: $\alpha_K \mapsto (\alpha_K)^x$ for $x \in \pi_0(G)$. This action preserves the equivalence classes of π-points, that is $\alpha_K \sim (\alpha_K)^x$ for any π-point α_K and any $x \in \pi_0(G)$. Moreover, the property of being a minimal representative is preserved by conjugation by elements of $\pi_0(G)$, and by extensions of scalars.

For a subgroup scheme $H \subset G$ we denote by $N_\tau(H)$ the stabilizer of H in $\tau = \pi_0(G)$.

Lemma 4.10. *Let $\alpha_K : K[t]/t^p \to K((G^0)^E \times E) \to KG$, $\beta_L : L[t]/t^p \to L((G^0)^F \times F) \to LG$ be two equivalent π-points of G, both minimal in their equivalence classes.*

(1) There exists $x \in \tau$ such that $(\beta_L)^x$ factors through $L((G^0)^E \times E)$;

(2) If $E = F$, then there exists $y \in N_\tau(E)$ such that α_K and $(\beta_L)^y$ determine equivalent π-points of $(G^0)^E \times E$.

Proof. Arguing as in the last part of the proof of Theorem 4.6, we find an algebraically closed field Ω/k and a field automorphism $\sigma : \Omega \to \Omega$ such that $\alpha_\Omega \sim \beta_\Omega^\sigma$ as Ω-rational π-points of G_Ω. Since Galois action does not affect either $(G^0)^E \times E$ or the minimality assumption on β_L, we may assume that $\alpha_\Omega \sim \beta_\Omega$ as π-points of G_Ω. Extending scalars from k to Ω we may further assume that α, β are two equivalent k-rational π-points of G where k is algebraically closed; in other words, we may assume that α, β are p-points of G. Hence,

$$\ker\{\alpha^*\} = \ker\{\beta^*\}$$

where α^*, β^* are the corresponding maps on cohomology. Adjusting by a scalar if necessary we may further assume

$$\alpha^* = \beta^*.$$

Let

$$\alpha = i_E \circ \alpha' : k[t]/t^p \xrightarrow{\alpha'} k((G^0)^E \times E) \xrightarrow{i_E} kG$$

$$\beta = i_F \circ \beta' : k[t]/t^p \xrightarrow{\beta'} k((G^0)^F \times F) \xrightarrow{i_F} kG$$

where i_E (respectively, i_F) is the map on group algebras induced by the embedding of group schemes $(G^0)^E \times E \hookrightarrow G$ (respectively, $(G^0)^F \times F \hookrightarrow G$). Consider the compositions

$$\bar{\alpha} : k[t]/t^p \xrightarrow{\alpha'} k((G^0)^E \times E) \longrightarrow kE \longrightarrow k\tau$$

and

$$\bar{\beta} : k[t]/t^p \xrightarrow{\beta'} k((G^0)^F \times F) \longrightarrow kF \longrightarrow k\tau$$

Since $\alpha^* = \beta^*$, we get

$$\bar{\alpha}^* = \bar{\beta}^*.$$

First, assume that $\bar{\alpha}^*$ and thus $\bar{\beta}^*$ are trivial (or, equivalently, $\bar{\alpha}, \bar{\beta}$ are not flat). By Proposition [14, 4.1], $\alpha' \sim \alpha_1 \otimes c_1 + c_2 \otimes \alpha_2$ with α_1 a π-point of $(G^0)^E$ and α_2 a p-point of E. Since $\bar{\alpha}^* = 0$, we conclude that $(c_2\alpha_2)^* = 0$. Since any p-point is flat, α_2 induces a non-trivial map in cohomology (see [14, 2.3]). Thus, $c_2 = 0$. By the minimality of α, we conclude that E is trivial. Similarly, F must be trivial.

Next, assume that both $\bar{\alpha}$ and $\bar{\beta}$ are flat. Thus, $\bar{\alpha}$, $\bar{\beta}$ are well-defined p-points of $k\tau$. Another application of Proposition [14, 4.1] implies that $\bar{\alpha}$, $\bar{\beta}$ are minimal representatives of their equivalence class in $\mathrm{P}(\tau)$. Since $\bar{\alpha}^* = \bar{\beta}^*$, the Quillen stratification theorem for finite groups (see [14, 3.6] for the p-points version) implies that there exists an elementary abelian subgroup $H \subset \tau$ such that $[\bar{\alpha}] \in \mathrm{P}(H)/N_\tau(H) \in \mathrm{P}(\tau)$. Choose H to be a minimal such subgroup. Since $[\bar{\alpha}]$ is also in $\mathrm{P}(E)/N_\tau(E)$, the Quillen stratification and the minimality of H imply that $H \subset E^{x_1}$ for some $x_1 \in \tau$. Since $[\bar{\alpha}] \in \mathrm{P}(H)/N_\tau(H)$, there exists a p-point $\gamma : k[t]/t^p \to kH \to k\tau$ such that $\bar{\alpha} \sim \gamma$ as π-points of τ. The minimality of the representative $\bar{\alpha}$ now implies that $H = E^{x_1}$. Similarly, $H = F^{x_2}$. Consequently, $\beta^{x_2 x_1^{-1}}$ factors through $(G^0)^E \times E$. This proves (1).

We now assume $\alpha, \beta : k[t]/t^p \to k(G^0 \rtimes E) \to kG$. (i.e. $E = F$). We essentially repeat a part of the proof of Theorem [14, 4.6] to complete the argument.

Let $G' = G^0 \rtimes E$. Let $j_E : (G^0)^E \times E \hookrightarrow G'$, $i : kG' \to kG$, and $p : kG' \to kE$ be the maps on group algebras induced by the embeddings $(G^0)^E \hookrightarrow G'$, $G' \hookrightarrow G$, and the projection $G' \twoheadrightarrow E$ respectively. We have the following factorizations for α and β:

$$\alpha = i \circ j_E \circ \alpha' : k[t]/t^p \xrightarrow{\alpha'} k((G^0)^E \times E) \xrightarrow{j_E} kG' \xrightarrow{i} kG$$

$$\beta = i \circ j_E \circ \beta' : k[t]/t^p \xrightarrow{\beta'} k((G^0)^E \times E) \xrightarrow{j_E} kG' \xrightarrow{i} kG$$

Recall the elements $\sigma_E \in \mathrm{H}^\bullet(E, k)$ and $\sigma_{G'} = p^*(\sigma_E) \in \mathrm{H}^\bullet(G', k)$ (cf. [14, 4.3]). Minimality of E and the bijection $\mathrm{P}(E) \simeq \mathrm{Proj}\,|E|$ imply that

$$(j_E \circ \alpha')^*(\sigma_{G'}) = (j_E \circ \alpha')^*(p^*(\sigma_E)) = (p \circ j_E \circ \alpha')^*(\sigma_E) \neq 0.$$

Thus, $\ker(j_E \circ \alpha')^*$ (and, similarly, $\ker(j_E \circ \beta')^*$) belongs to the open subvariety $\mathrm{Proj}\,\mathrm{H}^\bullet(G', k)[\sigma_G^{-1}] \subset \mathrm{Proj}\,\mathrm{H}^\bullet(G', k)$. Since $\alpha^* = \beta^*$, Corollary [14, 4.4] implies that there exists $y \in N_\tau(E) = N_\tau((G^0)^E \times E)$ such that

$$(j_E \circ \alpha')^* = ((j_E \circ \beta')^*)^y = (j_E \circ (\beta')^y)^*.$$

Since the map

$$j_E : \text{ k-rat'l pts of } \mathrm{Proj}\,\mathrm{H}^\bullet((G^0)^E \times E, k) \;\to\; \text{ k-rat'l pts of } \mathrm{Proj}\,\mathrm{H}^\bullet(G', k)$$

is an embedding by Lemma [14, 4.5], $(\alpha')^* = ((\beta')^y)^*$ so that $\alpha' \sim (\beta')^y \in \mathrm{P}((G^0)^E \times E)$ (by Theorem 3.6, for example). □

For each elementary abelian p-subgroup $E \subset \tau$, define $\Pi_0((G^0)^E \times E) \subset \Pi((G^0)^E \times E)$ to be the subspace of those π-points which do not admit a representative factoring through $(G^0)^E \times E'$ with E' a proper subgroup of E. Since each $\Pi((G^0)^E \times E') \to \Pi((G^0)^E \times E)$ is a closed map because $\mathrm{Proj}\,\mathrm{H}^\bullet((G^0)^E \times E'), k) \to \mathrm{Proj}\,\mathrm{H}^\bullet((G^0)^E \times E), k)$ is proper, $\Pi_0((G^0)^E \times E)$ is open in $\Pi((G^0)^E \times E)$.

Similarly, let $\Pi_0(G, E) \subset \Pi(G)$ be the locally closed subspace of equivalence classes of π-points which admit a representative factoring through $(G^0)^E \times E$ but not a representative factoring through $(G^0)^E \times E'$ for any E' a proper subgroup of E. Since conjugation by an element of τ does not affect the equivalence class of a π-point of G, we get a natural continuous map

(4.10.1) $$\theta_E : \Pi_0((G^0)^E \times E)/N_\tau(E) \to \Pi_0(G, E)$$

The following lemma shows that this map is a homeomorphism.

Lemma 4.11. (1) *Let E, F be two nonconjugate elementary abelian p-subgroups of τ. Then $\Pi_0(G,E) \cap \Pi_0(G,F) = \emptyset$.*

(2) *The map $\theta_E : \Pi_0((G^0)^E \times E)/N_\tau(E) \to \Pi_0(G,E)$ of (4.10.1) is a homeomorphism.*

Proof. (1). Suppose $\Pi_0(G,E) \cap \Pi_0(G,F) \neq \emptyset$. Then there exist π-points $\alpha_K : K[t]/t^p \to K((G^0)^E \times E) \to KG$, $\beta_L : L[t]/t^p \to L((G^0)^F \times F) \to LG$ such that $\alpha_K \sim \beta_L$ and both α_K, β_L are minimal representatives for their respective equivalence classes. By Lemma 4.10 we can find $x \in \tau$ such that $(\beta_L)^x$ factors through $G^0 \rtimes E$. Since β_L is a minimal representative, so is $(\beta_L)^x$. Since $(\beta_L)^x$ also factors through $G^0 \rtimes F^x$, it must factor through $G^0 \rtimes (E \cap F^x)$. Minimality of β_L^x now implies that $E \cap F^x = E = F^x$. Thus, E and F are conjugate.

(2) Surjectivity of θ_E is immediate from our definitions. To show the map is injective, consider the embedding $i : (G^0)^E \times E \hookrightarrow G$, and let α'_K, β'_L be two π-points of $(G^0)^E \times E$ such that $i \circ \alpha'_K \sim i \circ \beta'_L$. Lemma 4.10 implies that there exists $y \in N_\tau((G^0)^E \times E)$ such that $\alpha'_K \sim (\beta'_L)^y$, i.e. $[\alpha'_K] = [\beta'_L]^y$ in $\Pi_0((G^0)^E \times E)$. Thus, θ_E is injective. Continuity of θ_E^{-1} is immediate from the fact that (4.10.1) is a closed map (because $\Pi((G^0)^E \times E) \to \Pi(G)$ is a closed map). □

Proposition 4.12. *Let G be a finite group scheme of the form $G^0 \rtimes \tau$, with $\tau = \pi_0(G)$ and G^0 geometrically connected. Then there is a locally closed decomposition of $\Pi(G)$,*

$$\coprod \Pi_0((G^0)^E \times E)/N_\tau((G^0)^E \times E) \;\simeq\; \Pi(G)$$

where the disjoint union is indexed by conjugacy classes of elementary abelian p-subgroups of τ.

Proof. By Proposition [14, 4.2], any π-point admits a representative which factors through a subgroup scheme of the form $\mathbb{G}_{a(r),K} \times E \subset G_K$. Any such subgroup scheme embeds into a subgroup scheme of G_K of the form $((G^0)^E \times E)_K$. Thus, $\Pi(G) = \bigcup \Pi_0(G,E)$. The statement now follows from Lemma 4.11.

□

Example 4.13. The reader may find the following computation for $G = GL(3,\mathbb{F}_p)$ instructive, since there are distinct conjugacy classes of maximal elementary abelian p-groups in G. Assume $p \geq 3$. Consider the elements

$$e_{12} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ e_{13} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, e_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, e_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the subgroups generated by $(e_{12}, e_{13}), (e_{13}, e_{23}), (e_3, e_{13})$ represent the three distinct conjugacy classes of maximal elementary abelian p-groups in G.

Quillen's "stratification theorem" [24] implies that Spec $\mathrm{H}^\bullet(G,k)$ is the union of three irreducible surfaces, each the quotient of affine 2-space modulo a finite group, with common intersection an affine line modulo a finite group. Hence, Theorem 3.6 implies that $\Pi(G)$ is the 1-point union of 3 irreducible projective curves. In particular, any π-point of G is a specialization of one of the following three "generic" π-points:

$$\alpha_{k(z)} : k(z)[t]/(t^p) \to k(z)G, \quad t \mapsto z(e_{12}-1) + (e_{13}-1),$$

$$\beta_{k(z)} : k(z)[t]/(t^p) \to k(z)G, \quad t \mapsto z(e_{23} - 1) + (e_{13} - 1),$$

and

$$\gamma_{k(z)} : k(z)[t]/(t^p) \to k(z)G, \quad t \mapsto z(e_3 - 1) + (e_{13} - 1).$$

We conclude this section with another interesting family of examples.

Example 4.14. Let F be a finite field of characteristic $\ell \neq p$ with the property that F contains all p-th roots of unity. Then Quillen determines $\mathrm{H}^*(GL(n,F),k)$ in [25], establishing that

$$\mathrm{H}^*(GL(n,F),k) = (\mathrm{H}^*(T(n,F),k))^{\Sigma_n},$$

the invariants of the cohomology of the maximal torus $T(n,F) = (F^\times)^n$ under the permutation action of the symmetric group Σ_n. Thus,

$$\operatorname{Proj} \mathrm{H}^\bullet(GL(n,F),k) \ = \ \mathbb{P}^{n-1},$$

$n-1$ dimensional projective space over k.

Choose an element $1 \neq \mu \in F$ with the property that $\mu^p = 1$ and let $D_{i,i}(\mu) \in T(n,F)$ denote the diagonal matrix whose (i,i)-entry is μ and all of whose other diagonal entries equal 1. Let $K = k(\lambda_1, \dots, \lambda_n)$ denote the pure transcendental field extension of transcendence degree n over k and consider

$$\alpha_K : K[t]/t^p \to KT(n,F), \quad t \mapsto \sum_{i=1}^{n} \lambda_i (D_{i,i}(\mu) - Id).$$

Then the composition of α_K with the map of group algebras induced by $i : T(n,F) \to GL(n,F)$ represents a generic π-point of $GL(n,F)$. The composition $i \circ \alpha_K$ can be represented more efficiently by the equivalent π-point

$$\beta_L : L[t]/t^p \to LGL(n,F), \quad t \mapsto (\sum_{i=1}^{n-1} \sigma_i \cdot (D_{i,i}(\mu) - Id)) + (D_{n,n}(\mu) - Id)$$

where

$$L = k(\frac{\sigma_1}{\sigma_n}, \dots, \frac{\sigma_{n-1}}{\sigma_n})$$

and σ_i is the i-th elementary symmetric function in $\lambda_1, \dots \lambda_n$ (invariant under Σ_n).

5. The Π-support of an arbitrary G-module

One justification for considering the space $\Pi(G)$ of π-points of a finite group scheme G (rather than the simpler space $P(G)$ considered in [14]) is that this space serves as a useful invariant for kG-modules which are not necessarily finite dimensional. In particular, we shall verify in the next section (Corollary 6.7) that every subset of $\Pi(G)$ is the Π-support of some kG-module. Indeed, the consideration of non-closed points of $\Pi(G)$ when investigating infinite dimensional kG-modules is already foreshadowed in the work of Benson, Carlson, and Rickard (see [6]).

Theorem 4.6 allows us to extend the definition of the support to all, not necessarily finite dimensional, G-modules.

Definition 5.1. For a kG-module M, we define Π-support of M to be the subset

$$\Pi(G)_M \ \subset \ \Pi(G)$$

of those equivalence classes $[\alpha_K]$ of π-points such that $\alpha_K^*(M_K)$ is not projective for any representative $\alpha_K : K[t]/t^p \to KG$ of the equivalence class $[\alpha_K]$.

In view of Theorem 4.6, the properties of the π-support construction, $M \mapsto \Pi(G)_M$, stated in Propositions 3.2 and 3.3 extend to all kG-modules. The proofs of these properties for finite dimensional modules apply without change to infinite dimensional modules.

Proposition 5.2. *Let G be a finite group scheme over a field k and let M_1, M_2, M_3 be arbitrary kG-modules. Then*

(1) $\Pi(G)_k \ = \ \Pi(G)$.
(2) $\Pi(G)_{M_1 \otimes M_2} \ = \ \Pi(G)_{M_1} \cap \Pi(G)_{M_2}$.
(3) $\Pi(G)_{M_1 \oplus M_2} \ = \ \Pi(G)_{M_1} \cup \Pi(G)_{M_2}$.
(4) *If P is a projective kG-module, then $\Pi(G)_P = \emptyset$.*
(5) *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact, then*

$$\Pi(G)_{M_i} \subset \Pi(G)_{M_j} \cup \Pi(G)_{M_k}$$

where $\{i, j, k\}$ is any permutation of $\{1, 2, 3\}$.

We next extend the "projectivity test" given by support varieties to arbitrary kG-modules. This theorem is a measure of the non-triviality of our Π-support construction. One can view this as a statement that local projectivity implies projectivity. This result, generalizing a sequence of results by many authors, has its origins in L. Chouinard's proof [11] that projectivity of modules for a finite group G can be detected by restriction to elementary abelian p-subgroups $E \subset G$ and Dade's investigation [12] of modules for elementary abelian p-groups leading to the concept due to Carlson [8] of shifted subgroups of the group algebra kE.

Theorem 5.3. *Let G be a finite group scheme over a field k and let M be any kG-module. Then M is projective if and only if for any π-point $\alpha_K : K[t]/t^p \to KG$, $\alpha_K^*(M_K)$ is projective.*

Proof. By base change if necessary to the algebraic closure $\overline{k}$ of k, we may (and shall) assume that k is algebraically closed. The "only if" part is clear since π-points are flat maps. We assume that M satisfies the condition that $\alpha_K^* M_K$ is projective for every π-point $\alpha_K : K[t]/t^p \to KG$.

In the special case of a connected finite group scheme, the projectivity of M is given by [23, 2.2]. Let $j_{\mathcal{E}} : \mathcal{E} \to G$ be a quasi-elementary abelian subgroup scheme, so that $\mathcal{E} \simeq \mathbb{G}_{a(s)} \times E$ for some $s \geq 0$ and some elementary abelian group E of rank $r \geq 0$. Then $j_{\mathcal{E}}^*(M)$ satisfies the condition that $\beta_L^*(j_{\mathcal{E}}^* M)$ is projective for any π-point $\beta_L : L[t]/t^p \to L\mathcal{E}$. Choose an identification (as algebras, but not as Hopf algebras) of $k\mathcal{E}$ with $k\mathbb{G}_{a(r+s)}$. Since $\mathbb{G}_{a(r+s)}$ is connected, we conclude that $j_{\mathcal{E}}^* M$ is projective as a $k\mathbb{G}_{a(r+s)}$-module. Consequently, $j_{\mathcal{E}}^* M$ is projective as a $k\mathcal{E}$-module.

Consider the kG-module $\Lambda = \mathrm{End}_k(M)$. Observe that $j_{\mathcal{E}}^*(\Lambda) \simeq \mathrm{End}_k(j_{\mathcal{E}}^* M)$ as a $k\mathcal{E}$-module, and thus is projective. Therefore, $(j_{\mathcal{E}}^*(\Lambda))_K$ is projective for any field extension K/k. In particular, $\mathrm{H}^*(\mathcal{E}, j_{\mathcal{E}}^*(\Lambda)_K)$ vanishes in positive degrees for every $j_{\mathcal{E}} : \mathcal{E} \to G$ and every field extension K/k. By a theorem of Suslin [27], this implies that every homogeneous element of positive degree in $\mathrm{H}^*(G, \Lambda)$ is nilpotent.

To prove the projectivity of M, it suffices to prove for each irreducible kG-module S (necessarily finite dimensional) that $\mathrm{H}^i(G, S^\# \otimes M) = 0, i > 0$: this will then imply that $\mathrm{Hom}_G(S, \Omega^{-1} M) = \mathrm{Ext}_G^1(S, M) = 0$, and, hence, that $\Omega^{-1} M = 0$. This implies that M is injective and thus also projective since kG is a Frobenius algebra ([13]). Since $S^\# \otimes M$ necessarily satisfies $\alpha_K^*(S^\# \otimes M)$ is projective since $\alpha_K^*(M)$

is projective for any π-point α_K by Proposition 5.2, we may (and shall) simplify notation and replace $S^{\#} \otimes M$ by M.

Let G^0 denote the connected component of G, let $\tau = \pi_0(G)$ denote the discrete group of connected components of G. If $i : \mu \subset \tau$ is a subgroup and $G_\mu \subset G$ is the inverse image of μ with respect to the projection $G \to \tau$, then there is a natural transfer map $i_! : \mathrm{H}^*(G_\mu, M_{|G_\mu}) \to \mathrm{H}^*(G, M)$. A basic property of this transfer map guarantees that its composition with the natural map $i^* : \mathrm{H}^*(G, M) \to \mathrm{H}^*(G_\mu, M_{|G_\mu})$, $i_! \circ i^*$, equals multiplication by $[\tau : \mu]$, the index of μ in τ. Consequently, we may assume that τ is a finite p-group (one may consult [3] for a careful presentation of the transfer map in this situation).

We proceed by induction on the order of τ (the connected case already proved in [23, 2.2]) and consider some surjective map $\tau \to \mathbb{Z}/p$. Let G^1 denote the kernel of the composition $G \to \tau \to \mathbb{Z}/p$. By induction, we may assume that M is projective when restricted to G^1. Then the Lyndon-Hochschild-Serre spectral sequence for the extension $1 \to G^1 \to G \to \mathbb{Z}/p \to 1$ implies that

$$\mathrm{H}^*(G, M) \simeq \mathrm{H}^*(\mathbb{Z}/p, \mathrm{H}^0(G^1, M)). \tag{5.3.1}$$

Thus, to prove the vanishing of $\mathrm{H}^i(G, M), i > 0$, it suffices to verify that $\mathrm{H}^0(G^1, M)$ is projective as a $\mathbb{Z}/p$-module.

Assume to the contrary that $\mathrm{H}^0(G^1, M)$ is not projective as a $\mathbb{Z}/p$-module. Then no power of the generator T of $\mathrm{H}^\bullet(\mathbb{Z}/p, k) = k[T]$ acts trivially on $\mathrm{H}^*(\mathbb{Z}/p, \mathrm{H}^0(G^1, M))$, since the action of T induces the periodicity isomorphism $\mathrm{H}^n(\mathbb{Z}/p, M^{G_1}) \to \mathrm{H}^{n+2}(\mathbb{Z}/p, M^{G_1})$. The multiplicative structure of the Lyndon-Hochschild-Serre spectral sequence implies the compatibility of the pairing at the E_2-level with the pairing of abutments; in particular, we conclude the compatibility of the pairing

$$(E_2^{*,0}(k) = \mathrm{H}^*(\mathbb{Z}/p, \mathrm{H}^0(G^1, k))) \otimes (E_2^{*,0}(M) = \mathrm{H}^*(\mathbb{Z}/p, \mathrm{H}^0(G^1, M)))$$
$$\to (E_2^{*,0}(M) = \mathrm{H}^*(\mathbb{Z}/p, \mathrm{H}^0(G^1, M)))$$

via the edge homomorphism with the pairing

$$\mathrm{H}^\bullet(G, k) \otimes \mathrm{H}^*(G, M) \to \mathrm{H}^*(G, M).$$

Since the pairing at $E_2^{*,0}$ is that induced by the "identity" pairing

$$\mathrm{H}^0(G^1, k) \otimes \mathrm{H}^0(G^1, M) \to \mathrm{H}^0(G^1, M),$$

the isomorphism (5.3.1) implies that no power of the image of the generator via $\mathrm{H}^\bullet(\mathbb{Z}/p, \mathrm{H}^0(G^1, k)) \to \mathrm{H}^\bullet(G, k)$ acts trivially on $\mathrm{H}^*(G, M)$.

Since the action of $\mathrm{H}^*(G, k)$ on $\mathrm{H}^*(G, M)$ factors through $\mathrm{H}^*(G, \Lambda)$ (in other words, the action of $\mathrm{Ext}^*_G(k, k)$ on $\mathrm{Ext}^*_G(k, M)$ factors through $\mathrm{Ext}^*_G(M, M) = \mathrm{H}^*(G, \Lambda)$) and since we have shown that every element of $\mathrm{H}^*(G, \Lambda)$ is nilpotent, we obtain a contradiction. □

As mentioned above, we shall see in Corollary 6.7 that *any* subset of $\Pi(G)$ is of the form $\Pi(G)_M$ whereas $\mathrm{Proj}\, \mathrm{H}^\bullet(G, k)/ \mathrm{ann}_{\mathrm{H}^\bullet(G,k)} \mathrm{Ext}^*(M, M) \subset \mathrm{Proj}\, \mathrm{H}^\bullet(G, k)$ is always closed. However, the equality

$$\Pi(G)_M \;=\; \Psi_G^{-1}(\mathrm{Proj}(\mathrm{H}^\bullet(G, k)/ \mathrm{ann}_{\mathrm{H}^\bullet(G,k)}(\mathrm{Ext}^*_G(M, M)))$$

of Theorem 3.6 does admit the following partial generalization for arbitrary kG-modules.

Proposition 5.4. *Let G be a finite group scheme, and let M be a kG-module. Then*

$$\Psi_G(\Pi(G)_M) \subset \operatorname{Proj} \mathrm{H}^\bullet(G,k)/\operatorname{ann}_{\mathrm{H}^\bullet(G,k)}(\mathrm{Ext}^*_G(M,M)).$$

Proof. We must show that if $\alpha_K : K[t]/t^p \to KG$ is a π-point with the property that $\alpha_K^*(M_K)$ is not projective, then

$$\ker\{\alpha_K^*\} \cap \mathrm{H}^\bullet(G,k) \supset \operatorname{ann}_{\mathrm{H}^\bullet(G,k)}(\mathrm{Ext}^*_G(M,M)). \tag{5.4.1}$$

The commutative diagram

$$\begin{array}{ccc} \mathrm{H}^\bullet(G,k) & \xrightarrow{\otimes M} & \mathrm{Ext}^*_G(M,M) \\ \downarrow{\scriptstyle \otimes K} & & \downarrow{\scriptstyle \otimes K} \\ \mathrm{H}^\bullet(G_K,K) & \xrightarrow{\otimes_K M_K} & \mathrm{Ext}^*_{G_K}(M_K,M_K) \end{array}$$

implies the inclusion

$$\operatorname{ann}_{\mathrm{H}^\bullet(G,k)}(\mathrm{Ext}^*_G(M,M)) \subset \operatorname{ann}_{\mathrm{H}^\bullet(G_K,K)}(\mathrm{Ext}^*_{G_K}(M_K,M_K)) \cap \mathrm{H}^\bullet(G,k),$$

Hence, we may assume $K = k$ and that k is algebraically closed. For notational simplicity, we write α for α_K.

Recall that α is equivalent to some p-point $i_* \circ \alpha' : k[t]/t^p \to k\mathcal{E} \to kG$ factoring through the group algebra of some quasi-elementary abelian subgroup $i : \mathcal{E} = \mathbb{G}_{a(s)} \times E \subset G$. Thinking of Ext-groups in terms of extensions one sees easily that the square in the following diagram of algebra homomorphisms is commutative:

$$\begin{array}{ccccc} \mathrm{H}^\bullet(G,k) & \xrightarrow{i^*} & \mathrm{H}^\bullet(\mathcal{E},k) & \xrightarrow{(\alpha')^*} & \mathrm{H}^\bullet(k[t]/t^p,k) \\ {\scriptstyle \otimes M}\downarrow & & {\scriptstyle \otimes M}\downarrow & & \\ \mathrm{Ext}^*_G(M,M) & \longrightarrow & \mathrm{Ext}^*_{\mathcal{E}}(M,M) & & \end{array}$$

Thus, a simple diagram chase tells us that if (5.4.1) is valid for α', then it is valid for α.

Thus, to prove (5.4.1), we may assume that $G = \mathcal{E} = \mathbb{G}_{a(s)} \times E$ is quasi-elementary. Since $\mathcal{E}$ is a unipotent abelian group scheme,

$$\operatorname{ann}_{\mathrm{H}^\bullet(\mathcal{E},k)}(\mathrm{Ext}^*_{\mathcal{E}}(M,M)) = \operatorname{ann}_{\mathrm{H}^\bullet(\mathcal{E},k)}(\mathrm{H}^*(\mathcal{E},M)).$$

Since $\operatorname{ann}_{\mathrm{H}^\bullet(\mathcal{E},k)}(\mathrm{H}^*(\mathcal{E},M)) \subset \mathrm{H}^\bullet(\mathcal{E},k)$ does not change if we change the coproduct of $\mathcal{E}$, we may replace $\mathcal{E}$ by a group scheme isomorphic to $\mathbb{G}_{a(1)}^{r+s}$ in order to verify (5.4.1) for $\mathcal{E}$. In this case, we may assume that $\alpha : k[t]/t^p \to k\mathcal{E}$ is a map of Hopf algebras.

Let $\Lambda = \mathrm{End}_k(M)$. Since α is a map of Hopf algebras, $\alpha^*\Lambda = \mathrm{End}_k(\alpha^*(M))$ as a $k[t]/t^p$-algebra. Consider the following commutative diagram, where the left and right vertical maps are maps of algebras:

$$\begin{array}{ccc} \mathrm{H}^\bullet(\mathcal{E},k) & \xrightarrow{\alpha^*} & \mathrm{H}^\bullet(k[t]/t^p,k) \\ \downarrow & & \downarrow \\ \mathrm{H}^*(\mathcal{E},\Lambda) & \longrightarrow & \mathrm{H}^*(k[t]/t^p,\alpha^*\Lambda). \end{array} \tag{5.4.2}$$

Since $\alpha^*(M)$ is not projective, $\mathrm{H}^*(k[t]/t^p,\alpha^*\Lambda) = \mathrm{Ext}^*_{k[t]/t^p}(\alpha^*(M),\alpha^*(M))$ is non-trivial in positive degrees. Consequently, the right vertical map of (5.4.2) must be injective since the multiplication by the image of the generator of $\mathrm{H}^\bullet(k[t]/t^p, k)$ induces the periodicity isomorphism on $\mathrm{H}^*(k[t]/t^p, \alpha^*\Lambda)$. Since $\mathrm{H}^*(\mathcal{E}, \Lambda) \cong \mathrm{Ext}^*_{\mathcal{E}}(M, M)$, the fact that the kernel of the left vertical arrow of (5.4.2) is contained in the kernel of the top arrow implies (5.4.1).

□

The following corollary is an elaboration of the "local projectivity test" (Theorem 5.3). Of course, we can not replace $\Pi(G)_M$ in Corollary 5.5 by $P(G)_M$ because any module M whose Π-support is non-empty but contains no p-points is not projective but satisfies $P(G)_M = \emptyset$.

Corollary 5.5. *Let G be a finite group scheme over a field k and M be a kG-module. The following are equivalent:*

(1) *M is projective,*
(2) *$\Pi(G)_M = \emptyset$,*
(3) $\operatorname{Proj} \mathrm{H}^\bullet(G, k)/\operatorname{ann}(\mathrm{Ext}^*_G(M, M)) \; = \; \emptyset$

Proof. Theorem 5.3 implies the equivalence of (1) and (2), (1) clearly implies (3), and to finish the cycle we note that (3) implies (2) by Proposition 5.4. □

Π-supports satisfy the following functoriality properties with respect to change of finite group scheme.

Proposition 5.6. *Let $f : G' \to G$ be a flat map of finite group schemes over a field k. Then for any kG-module M,*

$$\Pi(G')_{f^*M} \;=\; (f_*)^{-1}(\Pi(G)_M).$$

Let $\rho : \Pi(G_K) \to \Pi(G)$ be the map induced by a field extension K/k (as in Corollary 2.8). Then for any kG-module M,

$$\Pi(G_K)_{M_K} \;=\; \rho^{-1}(\Pi(G)_M).$$

Furthermore, for any G_K-module N and any k-rational π-point $\alpha_k : k[t]/t^p \to kG$, $(K \otimes_k \alpha_k)^*(N)$ is free if and only if $\alpha_k^*(N_{|G_k})$ is free.

Proof. Let $\alpha_L : L[t]/t^p \to LG$ be a π-point of G. Then for a flat map $f : G' \to G$, $[\alpha_L] \in \Pi(G)_{f^*M}$ if and only if $\alpha_L^*((f^*M)_L) = (f \circ \alpha_L)^*(M_L)$ is not projective if and only if $[\alpha_L] \in (f_*)^{-1}(\Pi(G')_M)$.

The second claim follows immediately from the fact that the map ρ is induced by the identity map on π-points of G defined over field extensions $L/K/k$. Namely, for such a π-point $\alpha_L : L[t]/t^p \to LG$ and a kG-module M, we have $[\alpha_L] \in \Pi(G_K)_{M_K}$ if and only if $\alpha_L^*(M_L)$ is not projective if and only if $[\alpha_L] \in \Pi(G)_M$.

For the last assertion, observe that $(K \otimes_k \alpha_k)(1 \otimes t) = \alpha_k(t)$ is a K-linear endomorphism of N. The freeness of N as either a $K \otimes_k k[t]/t^p$ or $k[t]/t^p$-module is equivalent the non-existence of some $n \in N$ with $tn = 0$ and n not in the image of $t^{p-1} : N \to N$ (using t to also denote $1 \otimes t$). □

The last assertion of Proposition 5.6 enables us to construct very explicit (but necessarily infinite dimensional) examples of G-modules with no closed points in their support.

Example 5.7. Take k to be algebraically closed and let K/k be a non-trivial field extension. Consider any finite group scheme G over k such that $\Pi(G)$ has dimension bigger than 0 and consider any K-rational point $[\alpha_K] \in \Pi(G_K)$ which maps to a non-closed point of $\Pi(G)$. Let N be a finite dimensional G_K-module with $\Pi(G_K)_N = \{[\alpha_K]\}$. Then the restriction of N to G, $N_{|G}$, is not projective but has the property that $\Pi(G)_{N_{|G}}$ contains no closed points of $\Pi(G)$.

One indication of the potential usefulness of the Π-support of a G-module M is that its dimension has a representation-theoretic interpretation. If M is finite dimensional, then the following proposition asserts that the closed subset $\Pi(G)_M$ has (Krull) dimension equal to the "complexity" of M (cf. [1]). If M is not finite dimensional, then $\Pi(G)_M \subset \Pi(G)$ need not be closed. Following [22], we define the *subset dimension* of $W \subset \Pi(G)$ as

$$\mathrm{s.dim}(W) \stackrel{def}{=} \max_{s \in W} \dim(\overline{s}).$$

where $\overline{s}$ denotes the closure of an arbitrary point $s \subset \Pi(G)$. As in [5], we define the complexity of an arbitrary kG-module M to be the smallest c such that M can be realized as a filtered colimit of finite-dimensional modules of complexity c.

Proposition 5.8. *Let G be a finite group scheme over a field k. Then for any kG-module M, the "subset dimension" of $\Pi(G)_M$ equals the complexity of M.*

Proof. This is proved exactly as in [22, 3.17], and we leave the transcription to the interested reader. □

6. Tensor-ideal, thick subcategories of stmod (G)

In this section, we prove (in Theorem 6.3) the conjecture of Hovey, Palmieri, and Strickland [20] inspired by constructions of Benson, Carlson, and Rickard [7] for finite groups. In addition to the case of finite groups verified by [7], some special cases of Theorem 6.3 were proved by Hovey and Palmieri in [18], [19]. We also give an alternative description of the Π-support $\Pi(G)_M$ of a kG-module following a construction of Benson, Carlson, and Rickard for finite groups [6]. As we have throughout this paper, we work in the context of an arbitrary finite group scheme G over an arbitrary field k.

Let G be a finite group scheme over a field k. Recall that the stable module category StMod (G) is the category whose objects are kG-modules, and whose group of homomorphisms between two kG-modules M, N is given by the following quotient:

$$\mathrm{Hom}_G(M,N)/\{f : M \to N \text{ factoring through some projective}\}.$$

So defined, StMod (G) is a triangulated category, with $M[1]$ represented by the cokernel of an embedding of M in an injective kG-module (i.e., $M[1] = \Omega^{-1}M$, where ΩM is the Heller shift of M, given as the kernel of a surjective map from a projective kG-module to M). Distinguished triangles come from short exact sequences in the abelian category of G-modules.

We denote by stmod $(G) \subset$ StMod (G) the (triangulated) full subcategory of StMod (G) whose objects are finite dimensional kG-modules. We shall say that kG-modules are *stably isomorphic* if they are isomorphic in StMod (G).

We recall that a full subcategory $\mathcal{C}$ of a triangulated category $\mathcal{T}$ is said to be a *thick* subcategory if it is triangulated, closed under direct summands, and closed under finite direct sums. Every thick subcategory of stmod (G) is obtained by restricting some thick subcategory of StMod (G) to its full subcategory of finite dimensional kG-modules. If $\mathcal{T}$ has suitable (tensor) products (i.e., is symmetric monoidal), then a triangulated subcategory $\mathcal{C} \subset \mathcal{T}$ is said to be *tensor-ideal* if it is closed under taking tensor products with any element in $\mathcal{T}$.

Example 6.1. Let $C \subset \Pi(G)$ be a subset and let $\mathcal{C}_C \subset \mathrm{stmod}\,(G)$ be the full subcategory of finite dimensional kG-modules M with $\Pi(G)_M \subset C$. Then Propositions 3.3 and 3.2 enable us to conclude that $\mathcal{C}_C$ is a thick, tensor-ideal subcategory of stmod (G).

Following Rickard [26], we associate to any thick, tensor-ideal subcategory $\mathcal{C} \subset$ stmod (G) two (infinite dimensional) modules E_C, F_C defined up to natural isomorphism with the following properties. Although these properties are stated for finite groups in [26] (cf. also [22] for connected finite group schemes), the proofs apply to any finite group scheme.

Proposition 6.2. *Let G be a finite group scheme over a field k. For each thick, tensor-ideal subcategory $\mathcal{C} \subset stmod\,(G)$ let $E_{\mathcal{C}}, F_{\mathcal{C}} \in StMod(G)$ denote the Rickard idempotents associated to $\mathcal{C}$ as constructed in* [26]*. Then*

1. *$E_{\mathcal{C}}, F_{\mathcal{C}}$ fit in a distinguished triangle in StMod (G)*
$$E_{\mathcal{C}} \to k \to F_{\mathcal{C}} \to E_{\mathcal{C}}[1].$$
2. *$E_{\mathcal{C}}$ is a filtered colimit of modules from $\mathcal{C}$ and $F_{\mathcal{C}}$ is $\mathcal{C}$-local (i.e. there are no non-trivial maps $M \to F_{\mathcal{C}}$ in StMod (G) whenever $M \in \mathcal{C}$).*
3. *For any $M \in stmod\,(G)$, $M \in \mathcal{C}$ if and only if M is stably isomorphic to $E_{\mathcal{C}} \otimes M$ if and only if $F_{\mathcal{C}} \otimes M$ is projective.*
4. *$E_{\mathcal{C}} \otimes E_{\mathcal{C}}$ is stably isomorphic to $E_{\mathcal{C}}$, $E_{\mathcal{C}} \otimes F_{\mathcal{C}}$ is projective, and $F_{\mathcal{C}} \otimes F_{\mathcal{C}}$ is stably isomorphic to $F_{\mathcal{C}}$.*

A subset $W \subset \Pi(G)$ is *closed under specialization* if for any equivalence class of π-points $[\alpha] \in W$, W also contains the equivalence class of every specialization of α. Equivalently, W is closed under specialization if whenever a point lies in W then the closure of the point is contained in W. The following theorem gives a bijective correspondence between subsets of $\Pi(G)$ closed under specialization and thick tensor-ideal subcategories of stmod (G). Since this correspondence clearly respects inclusions of subsets and subcategories, one could phrase the following theorem more elaborately in terms of lattices. This is the form in which Hovey-Palmieri-Strickland phrase their conjecture, which we now prove.

Observe that our proof of Theorem 6.3 requires in an essential way our consideration of arbitrary kG-modules and the properties given in Proposition 5.2.

Theorem 6.3. (Hovey-Palmieri-Strickland Conjecture) *Let G be a finite group scheme over a field k. Then there is a natural bijection between the subsets $W \subset \Pi(G)$ which are closed under specialization and the thick, tensor-ideal subcategories $\mathcal{C}$ of stmod (G).*

Namely, we associate to any subset $W \subset \Pi(G)$ the thick, tensor-ideal category $\mathcal{C}_W \subset stmod(G)$ of all finite dimensional modules M with $\Pi(G)_M \subset W$,
$$W \ \mapsto \ \mathcal{C}_W.$$
Moreover, we associate to any full subcategory $\mathcal{C} \subset stmod\,(G)$ the subset $W_{\mathcal{C}} \equiv \cup_{M \in Obj(\mathcal{C})} \Pi(G)_M$ closed under specialization,
$$\mathcal{C} \ \mapsto \ W_{\mathcal{C}}.$$

These constructions are mutually inverse when restricted to subsets $W \subset \Pi(G)$ closed under specialization and thick, tensor-ideal subcategories $\mathcal{C}$ of stmod(G).

Proof. For any $W \subset \Pi(G)$, $\mathcal{C}_W \subset \mathrm{stmod}\,(G)$ is a thick, tensor-ideal category by Proposition 3.3 and Proposition 3.2. Moreover, if $\mathcal{C} \subset \mathrm{stmod}\,(G)$ is a full subcategory, then the subset $\cup_{M \in Obj(\mathcal{C})}\Pi(G)_M \subset \Pi(G)$ is closed under specialization. We proceed to show that these correspondences are mutually inverse, using the Rickard idempotents of Proposition 6.2.

We first prove for any $W \subset \Pi(G)$ closed under specialization that $W = W_{\mathcal{C}_W}$. Essentially by definition, we have the containment $W_{\mathcal{C}_W} \subset W$ for any W. Conversely, any W closed under specialization is a (not necessarily finite) union of closed subsets, $W = \cup_i C_i$. By Proposition 3.4, we may find finite dimensional modules $M_{C_i} \in \mathcal{C}_W$ with $\Pi(G)_{M_{C_i}} = C_i$ so that $C_i \subset W_{\mathcal{C}_W}$, and thus $W = \cup_i C_i \subset W_{\mathcal{C}_W}$.

To complete the proof of the theorem, we show for any tensor-ideal thick subcategory $\mathcal{C} \subset \mathrm{stmod}\,(G)$ that $\mathcal{C}_{W_\mathcal{C}} = \mathcal{C}$. Once again, one inclusion, namely $\mathcal{C} \subset \mathcal{C}_{W_\mathcal{C}}$, holds essentially by definition. To show the opposite inclusion $\mathcal{C}_{W_\mathcal{C}} \subset \mathcal{C}$ we first observe that $\Pi(G)_M \cap \Pi(G)_{F_\mathcal{C}} = \emptyset$ for any $M \in \mathcal{C}$ since $M \otimes F_\mathcal{C}$ is projective. Since $W_\mathcal{C} = \bigcup_{M \in Obj(\mathcal{C})} \Pi(G)_M$, we conclude that

$$W_\mathcal{C} \cap \Pi(G)_{F_\mathcal{C}} \;=\; \emptyset$$

Now let $M \in \mathcal{C}_{W_\mathcal{C}}$, that is $\Pi(G)_M \subset W_\mathcal{C}$. Then $\Pi(G)_M \cap \Pi(G)_{F_\mathcal{C}} \subset W_\mathcal{C} \cap \Pi(G)_{F_\mathcal{C}} = \emptyset$. Hence, $\Pi(G)_{M \otimes F_\mathcal{C}} = \emptyset$, so that $M \otimes F_\mathcal{C}$ is projective and thus $M \in C$. □

As a corollary of Theorem 6.3 and a theorem of R. Thomason, we get the following suggestive bijection.

Corollary 6.4. *Let G be a finite group scheme over a field k of positive characteristic. Let $\mathrm{D^{perf}}(\mathrm{Proj}\,\mathrm{H}^\bullet(\mathrm{G},\mathrm{k}))$ be the full subcategory of perfect complexes in the derived category of coherent $\mathcal{O}_{\mathrm{Proj}\,\mathrm{H}^\bullet(G,k)}$-modules, a tensor, triangulated category. Then there is an isomorphism between the lattice of thick, tensor-ideal subcategories of stmod(G) and the lattice of thick, tensor-ideal subcategories of $\mathrm{D^{perf}}(\mathrm{Proj}\,\mathrm{H}^\bullet(\mathrm{G},\mathrm{k}))$.*

Proof. Theorem 6.3 establishes a bijection between the lattice of thick, tensor-ideal subcategories of stmod (G) and the lattice of subsets of $\Pi(G)$ which are closed under specialization whereas Thomason [30, 3.15] establishes a bijection between the latter lattice and the lattice of thick, tensor-ideal subcategories of $\mathrm{D^{perf}}(\mathrm{Proj}\,\mathrm{H}^\bullet(\mathrm{G},\mathrm{k}))$. □

The "Rickard idempotents" of Proposition 6.2 enable us to realize *any* subset $S \subset \Pi(G)$ as the Π-support of some kG-module.

Definition 6.5. Let G be a finite group scheme over a field k of characteristic $p > 0$. For each equivalence class $[\alpha] \in \Pi(G)$, let $E_{[\alpha]}, F_{[\alpha]}$ be the Rickard idempotents associated to the thick, tensor-ideal subcategory $\mathcal{C}_{[\alpha]} \subset \mathrm{stmod}\,(G)$ consisting of finite dimensional kG-modules whose Π-supports are contained in the closure of $[\alpha]$. Let $\widetilde{E}_{[\alpha]}, \widetilde{F}_{[\alpha]}$ be the Rickard idempotents associated to the thick, tensor-ideal subcategory $\widetilde{\mathcal{C}}_{[\alpha]} \subset \mathrm{stmod}\,(G)$ consisting of finite dimensional kG-modules whose Π-supports are strictly contained in the closure of $[\alpha] \in \Pi(G)$ (i.e., do not contain $[\alpha]$). Finally, set

$$\kappa_{[\alpha]} \;\equiv\; E_{[\alpha]} \otimes \widetilde{F}_{[\alpha]}.$$

Proposition 6.6. *Let G be a finite group scheme over a field k, let $[\alpha] \in \Pi(G)$ be an equivalence class of π-points of G, and let $E_{[\alpha]}, F_{[\alpha]}, \kappa_{[\alpha]}$ be the kG-modules defined above. Then*

(1) *The Π-support of $E_{[\alpha]}$ is the closure of $[\alpha] \in \Pi(G)$.*
(2) *The Π-support of $F_{[\alpha]}$ is the complement in $\Pi(G)$ of the closure of $[\alpha]$.*
(3) *The Π-support of $\kappa_{[\alpha]}$ equals $\{[\alpha]\}$.*

Proof. We first show for any closed under specialization subset $W \subset \Pi(G)$ with associated tensor-ideal thick subcategory $\mathcal{C} = \mathcal{C}_W$ that $\Pi(G)_{E_\mathcal{C}} = W$ and $\Pi(G)_{F_\mathcal{C}}$ is the complement of W.

Since W is closed under specialization, $W = \bigcup V_i$ where V_i are closed subsets of $\Pi(G)$. Let M_{V_i} be a finite dimensional kG-module with Π-support V_i. Since $M_{V_i} \otimes E_\mathcal{C}$ is stably isomorphic to M_{V_i}, the tensor product property implies the inclusion

$$V_i = \Pi(G)_{M_{V_i}} \subset \Pi(G)_{E_\mathcal{C}}.$$

Thus, $W \subset \Pi(G)_{E_\mathcal{C}}$. To prove the opposite inclusion, pick a π-point β which is not in W. Applying Proposition 6.2.2, we write $E_\mathcal{C} = \operatorname{colim} M_i$ as a filtered colimit of finite dimensional modules M_i such that $\Pi(G)_{M_i} \subset W$. Since $\beta \notin W$, we conclude that $\beta^*(M_i)$ is projective for all M_i. Since the colimit of injectives is injective and since a KG-module is projective if and only if it is injective ([13]), we conclude that $\beta^*(E_\mathcal{C})$ is also projective. Thus, $[\beta] \notin \Pi(G)_{E_\mathcal{C}}$ and the inclusion

$$\Pi(G)_{E_\mathcal{C}} \subset W$$

follows.

Since $E_\mathcal{C} \otimes F_\mathcal{C}$ is projective, Proposition 3.2 implies that $\Pi(G)_{E_\mathcal{C}} \cap \Pi(G)_{F_\mathcal{C}} = \emptyset$ and thus $\Pi(G)_{F_\mathcal{C}}$ is contained in the complement of W. On the other hand, Proposition 3.3 together with Proposition 6.2.1, imply the equality

$$\Pi(G)_{E_\mathcal{C}} \cup \Pi(G)_{F_\mathcal{C}} = \Pi(G).$$

Thus, $\Pi(G)_{F_\mathcal{C}}$ is precisely the complement of W.

Now, (1) and (2) follow by applying the above to $W = \overline{[\alpha]}$, the closure $\{[\alpha]\} \subset \Pi(G)$. Applying the above argument to $W = \overline{[\alpha]} - [\alpha]$ in order to determine $\Pi(G)_{\widetilde{F}_{[\alpha]}}$ and using Proposition 3.2 again, we conclude (3). □

The following is an immediate corollary of Proposition 6.6 together with Proposition 5.2(3).

Corollary 6.7. *Let G be a finite group scheme over a field k. Then for any subset $S \subset \Pi(G)$, there exists some kG-module M_S with Π-support equal to S,*

$$\Pi(G)_{M_S} = S.$$

Namely, we may take

$$M_S = \bigoplus_{[\alpha] \in S} \kappa_{[\alpha]}.$$

Using κ-modules, one can provide an equivalent characterization of the Π-support of a kG-module. This is an interpretation using π-points of the definition of Benson, Carlson, Rickard [6] of the support variety of an infinite dimensional module (for a finite group).

Proposition 6.8. *For any finite group scheme G over a field k and any equivalence class of π-points $[\alpha] \in \Pi(G)$,*

$$\Pi(G)_M = \{[\alpha] : \kappa_{[\alpha]} \otimes M \text{ is not projective }\}.$$

Proof. By Theorem 5.3, $\kappa_{[\alpha]} \otimes M$ is not projective if and only if the Π-support of $\kappa_{[\alpha]} \otimes M$ is non-empty which by Proposition 3.2 is the case if and only the Π-supports of $\kappa_{[\alpha]}$ and M have non-empty intersection. Since $\Pi(G)_{\kappa_{[\alpha]}} = \{[\alpha]\}$ by Proposition 6.6.3, this is the case if and only if $[\alpha] \in \Pi(G)_M$. □

Our final proposition verifies that the action on $\Pi(G)$ by an automorphism of k/k' constructed in Proposition 4.4 naturally determines an action on $\Pi(G)_M$ provided that the kG-module M is obtained by base change from a G'-module where $G = G' \times_{\operatorname{Spec} k'} \operatorname{Spec} k$. The existence of such an action is therefore an obstruction to descending the kG-module structure on M to a $k'G'$-module structure.

Proposition 6.9. *Let k/k' be a field extension and $\sigma : k \to k$ a field automorphism over k'. Assume that the finite group scheme G over k is defined over k', so that $G = G' \times_{\operatorname{Spec} k'} \operatorname{Spec} k$.*

(1) If M is a kG-module defined over k', then the action of σ stabilizes $\Pi(G)_M$.

(2) If k/k' is a finite Galois extension with Galois group τ and if C is a subset of $\Pi(G)$ of the form $\Pi(G)_M$ for some kG-module M, then there exists a $k'G'$-module N with the property that $\Pi(G)_{N_k}$ is the closure of C under the action of τ. If C is closed, we may choose N to be finite dimensional.

Proof. The first statement follows immediately from the second part of Proposition 4.4.

We now assume that k/k' is Galois. If V is a k-vector space and if $\sigma \in \tau$, we define a new k-vector space V^σ by

$$V^\sigma \equiv k \otimes_\sigma V,$$

where the tensor product $k \otimes_\sigma V$ is taken by viewing k as a k-module via σ. Equivalently, V coincides with V^σ as an abelian group but the action of k is twisted by σ^{-1}: $a \circ (1 \otimes_\sigma v) = a \otimes_\sigma v = 1 \otimes_\sigma \sigma^{-1}(a)v$. Since the group G is defined over k', the algebra $kG = k \otimes_{k'} k'G'$ can be naturally identified with $kG^\sigma = k \otimes_\sigma k \otimes_{k'} k'G'$ via the k-algebra isomorphism

$$kG = k \otimes_{k'} k'G' \simeq k \otimes_\sigma k \otimes_{k'} k'G' = kG^\sigma \tag{6.9.1}$$

$$a \otimes f \mapsto a \otimes 1 \otimes f.$$

For a kG-module M, the twisted module M^σ has a natural structure of a kG^σ-module: $kG^\sigma \otimes M^\sigma = (kG \otimes M)^\sigma \to M^\sigma$. We consider M^σ as a G-module via the algebra identification 6.9.1.

Let $C = \Pi(G)_M$ for some kG-module M. Let $\widetilde{M} = k \otimes_{k'} (M_{|G'})$. There is an isomorphism of kG-modules

$$\widetilde{M} \simeq \bigoplus_{\sigma \in \tau} M^\sigma, \tag{6.9.2}$$

given explicitly by

$$a \otimes m \mapsto (a \otimes_\sigma m)_{\sigma \in \tau}.$$

Indeed, one readily observes that $k \otimes_{k'} k \to \bigoplus_{\sigma\in\tau} k^\sigma$ is a k-linear isomorphism: if $\{\alpha_\sigma\}_{\sigma\in\tau}$ is a basis of k over k', then the elements $(1 \otimes_\sigma \sigma'(\alpha_\sigma))_{\sigma\in\tau} \in \bigoplus_{\sigma\in\tau} k^\sigma$ indexed by $\sigma' \in \tau$, form a basis of $\bigoplus_{\sigma\in\tau} k^\sigma$ and are in the image of the map above. To verify isomorphism 6.9.2 for a general module M, we tensor $k \otimes_{k'} k \simeq \bigoplus_{\sigma\in\tau} k^\sigma$ with M and observe that $k^\sigma \otimes_k M = k \otimes_\sigma k \otimes_k M = k \otimes_\sigma M = M^\sigma$.

We proceed to verify that

$$(\Pi(G)_M)^\sigma = \Pi(G)_{M^{\sigma^{-1}}},$$

i.e. that for a π-point $\alpha_K : K[t]/t^p \to KG_K$, $\alpha_K^*((M^{\sigma^{-1}})_K)$ is projective if and only if $(\alpha_K^\sigma)^*(M_K)$ is projective. By enlarging the field K if necessary, we assume that σ extends to an automorphism of K which we denote by $\tilde\sigma$. Let $\alpha_K(t) = \sum_i a_i t_i$ where $a_i \in K$ and $\{t_i\}$ is a basis of the algebra $k'G'$ over k'. Then t acts on the $K[t]/t^p$ module $(\alpha_K^\sigma)^*(M_K)$ via $\alpha_K^\sigma(t) = \sum_i \tilde\sigma(a_i)t_i$. As the action of $\alpha_K(t) = \sum_i a_i t_i$ on $M_K^{\sigma^{-1}} = (M_K)^{\sigma^{-1}}$ is the same as the action of $\sum_i \tilde\sigma(a_i)t_i$ on M_K, we conclude the desired equality $(\Pi(G)_M)^\sigma = \Pi(G)_{M^{\sigma^{-1}}}$. Thus, isomorphism (6.9.2) implies that

$$\Pi(G)_{\widetilde{M}} \ = \ \bigcup_{\sigma\in\tau} (\Pi(G)_M)^\sigma.$$

Therefore, we have shown for $N = M_{|G'}$ that $\Pi(G)_{N_k} = \Pi(G)_{\widetilde{M}}$ is the closure of $C = \Pi(G)_M$ with respect to the action of τ. By definition, if C is closed, then M can be chosen to be finite dimensional, and, therefore, N will also be finite dimensional.

□

Corollary 6.7 implies that any subset of $\Pi(G)$ is realizable as a support set of some G-module M. If a subset is closed, then by definition it is realizable by a finite-dimensional module. Thus, the proposition above immediately implies the following "realization" result.

Corollary 6.10. *Let k/k' be a finite Galois field extension, and $C \subset \Pi(G)$ be a (closed) subset stable under the action of $Gal(k/k')$. Then there exists a (finite-dimensional) $k'G'$-module N such that $\Pi(G)_{N_k} = C$.*

7. Realization of the scheme structure for $\Pi(G)$

In this final section, we verify that we can endow the topological space $\Pi(G)$ with a sheaf of k-algebras determined by the stable module category $\mathrm{stmod}\,(G)$ so that the associated ringed space is isomorphic to the scheme $\mathrm{Proj}\,\mathrm{H}^\bullet(G,k)$.

As usual, G will denote a finite group scheme over a field k of positive characteristic. We shall frequently make the identification

$$\mathrm{H}^i(G,k) \simeq \mathrm{Hom}_G(\Omega^i k, k) \simeq \mathrm{Hom}_{\mathrm{stmod}\,(G)}(\Omega^{i+j}k, \Omega^j k),$$

and we shall use the same notation α for a cohomology class in $\mathrm{H}^i(G,k)$ and any G-map $\Omega^{i+j} \to \Omega^j k$ whose stable equivalence class represents this cohomology class.

We denote by $\mathcal{C} = \mathrm{stmod}\,(G)$ the stable module category, and by $\mathcal{C}_W$ the thick tensor ideal subcategory associated to a closed subset $W \subset \Pi(G)$ as in Theorem 6.3. We use the standard notation $\mathcal{C}/\mathcal{C}_W$ for the triangulated category obtained by localizing $\mathcal{C}$ with respect to $\mathcal{C}_W$. Thus, $\mathrm{Obj}(\mathcal{C}/\mathcal{C}_\mathrm{W}) = \mathrm{Obj}(\mathcal{C})$ and maps from M to N in $\mathcal{C}/\mathcal{C}_W$ are represented by triples $M \overset{s}{\leftarrow} Q \overset{f}{\rightarrow} N$, where the kernel and cokernel of s are objects of $\mathcal{C}_W$ (i.e., s is a $\mathcal{C}_W$ isomorphism).

We now define a sheaf of (not necessarily commutative) rings on $\Pi(G)$.

Definition 7.1. Consider the presheaf of k-algebras $\Theta_{\Pi(G)}$ on the topological space $\Pi(G)$ defined on the complement $(\Pi(G) - W)$ of a closed subset $W \subset \Pi(G)$ by

$$(\Pi(G) - W) \mapsto \mathrm{End}_{\mathcal{C}/\mathcal{C}_W}(k)$$

and whose restriction maps are the evident localization maps. Let $\widetilde{\Theta}_{\Pi(G)}$ be the associated sheaf.

Denote by $\mathcal{H}$ the projective scheme $\mathrm{Proj}\,\mathrm{H}^\bullet(G,k)$, and by $\mathcal{O}_\mathcal{H}$ the structure sheaf of $\mathcal{H}$. For $\zeta \in \mathrm{H}^n(G,k)$, where n is even if $p > 2$, let $V(\zeta) \subset \mathcal{H}$ be the hypersurface defined by the ideal generated by ζ. We have $\mathcal{O}_\mathcal{H}(\mathcal{H} - V(\zeta)) = (\mathrm{H}^\bullet(G,k)[\frac{1}{\zeta}])_0$, the degree zero part of the localization of the cohomology ring at ζ.

Next, we describe the map which will later serve to identify structure sheaves on $\mathcal{H}$ and $\Pi(G)$. The construction relies on a result of J. Carlson, P. Donovan, and W. Wheeler [10, 3.1] which is stated for finite groups but whose proof applies verbatim to any finite group scheme.

Let $W \subset \Pi(G)$ be a closed subset and let $U = \Pi(G) - W$. Identifying $\mathcal{H} \simeq \Pi(G)$ via the homeomorphism Ψ_G of 3.6, we may consider W, U as subsets of $\mathcal{H}$. By Proposition 3.7, Ψ_G identifies $\Pi(G)_{L_\zeta} \subset \Pi(G)$ with $V(\zeta) \subset \mathcal{H}$. We shall use notation W_ζ for both $\Pi(G)_{L_\zeta}$ and $V(\zeta)$. Let $k \overset{s}{\leftarrow} M \overset{\alpha}{\rightarrow} k \in \mathrm{End}_{\mathcal{C}/\mathcal{C}_W}(k)$. Since s is a $\mathcal{C}_W$-isomorphism, it fits into an exact sequence $0 \to N \to M \to k \to 0$ such that $\Pi(G)_N \subset W$. Let $\zeta \in \mathrm{H}^\bullet(G,k)$ be a homogeneous cohomology class of degree n such that $W \subset W_\zeta = \Pi(G)_{L_\zeta}$. By [10, 3.1], we may find $\gamma : \Omega^{nt}k \to M$ and a commutative diagram

$$\begin{array}{ccccc} k & \overset{\zeta^t}{\longleftarrow} & \Omega^{tn}k & \overset{\beta}{\longrightarrow} & k \\ \| & & \big\downarrow{\scriptstyle\gamma} & & \| \\ k & \underset{s}{\longleftarrow} & M & \underset{\alpha}{\longrightarrow} & k \end{array} \tag{7.1.1}$$

Thus, we can represent $k \overset{s}{\leftarrow} M \overset{\alpha}{\rightarrow} k$ as $k \overset{\zeta^t}{\leftarrow} \Omega^{tn}k \to k$ in $\mathrm{End}_{\mathcal{C}/\mathcal{C}_{W_\zeta}}(k)$.

We now define a map

$$\phi_W : \mathrm{End}_{\mathcal{C}/\mathcal{C}_W}(k) \to \mathcal{O}_\mathcal{H}(U) \tag{7.1.2}$$

for any open $U \subset \Pi(G)$. To define a regular function $\phi_W(k \leftarrow M \to k) \in \mathcal{O}_\mathcal{H}(U)$, it suffices to define it locally. Since the basic open sets of the form $U_\zeta = \mathcal{H} - W_\zeta$ form a basis of the topology on $\mathcal{H}$, it suffices to define the restrictions of $\phi_W(k \leftarrow M \to k)$ to open subsets $U_\zeta \subset U$. For this, we choose a representative of $k \leftarrow M \to k$ of the form $k \overset{\zeta^t}{\leftarrow} \Omega^{tn}k \overset{\beta}{\rightarrow} k$ and define

$$\phi_W(k \leftarrow M \to k) \downarrow_{U_\zeta} = \beta/\zeta^t.$$

In the following proposition we check that ϕ_W is well-defined. We remind the reader that we identify $\Pi(G)$ and $\mathcal{H}$ as topological spaces via the homeomorphism Ψ_G

Proposition 7.2. *Let $W \subset \Pi(G)$ be a closed subset, and let $U = \mathcal{H} - W$. The map ϕ_W of (7.1.2) is well-defined and determines a ring homomorphism*

$$\phi_W : \mathrm{End}_{\mathcal{C}/\mathcal{C}_W}(k) \to \mathcal{O}_{\mathcal{H}}(U).$$

Moreover, for an open subset $U' \subset U$ of $\Pi(G)$, we have a commutative diagram

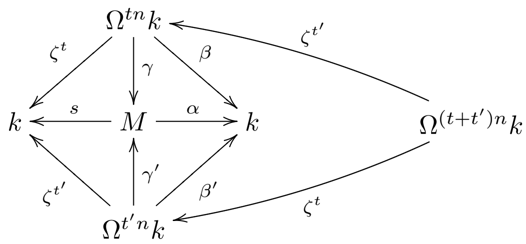
(7.2.1)
$$\begin{array}{ccc} \mathrm{End}_{\mathcal{C}/\mathcal{C}_W}(k) & \xrightarrow{\phi_W} & \mathcal{O}_{\mathcal{H}}(U) \\ \downarrow & & \downarrow \\ \mathrm{End}_{\mathcal{C}/\mathcal{C}_{W'}}(k) & \xrightarrow{\phi_{W'}} & \mathcal{O}_{\mathcal{H}}(U') \end{array}$$

where $W' = \Pi(G) - U'$ and the vertical maps are the natural restriction maps.

Proof. To show that ϕ_W is well-defined, we have to check that

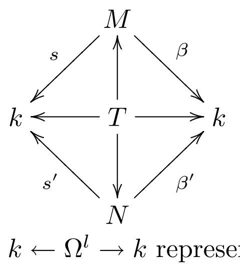
(1) ϕ_W does not depend on the choice of the commutative diagram 7.1.1 for a given $k \leftarrow M \to k$
(2) ϕ_W does not depend on the choice of representative $k \leftarrow M \to k$
(3) $\phi_W \downarrow_{U_\zeta}$ and $\phi_W \downarrow_{U_\xi}$ agree on the intersection $U_\zeta \cap U_\xi = U_{\zeta\xi}$

(1) follows by examining the commutative diagram

(7.2.2)


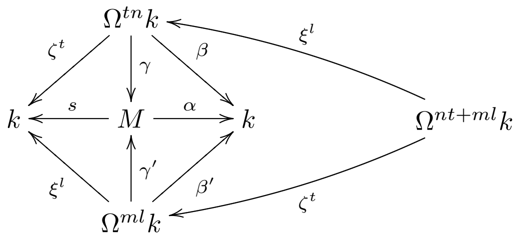
The diagram implies that, considered as cohomology classes, $\beta\zeta^{t'} = \beta'\zeta^t$. Thus, $\beta/\zeta^t = \beta'/\zeta^{t'}$ on U_ζ.

To show (2), observe that by definition of the equivalence relation on morphisms in $\mathcal{C}/\mathcal{C}_W$, $k \leftarrow M \to k$ and $k \leftarrow N \to k$ represent the same endomorphism if and only if there is a commutative diagram

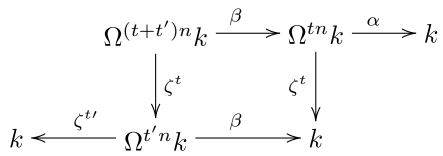


By choosing the endomorphism $k \leftarrow \Omega^l \to k$ representing $k \leftarrow T \to k$ as in (7.2.1), we conclude that it also represents both $k \leftarrow M \to k$ and $k \leftarrow N \to k$. This verifies (2).

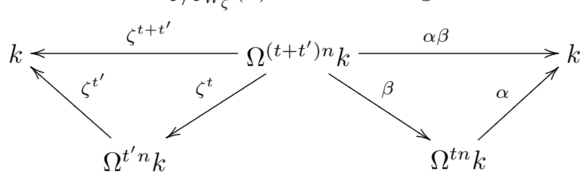
To prove (3), one proceeds exactly as for (1) provided one replaces diagram (7.2.2) by the following diagram



The additivity of ϕ_W is evident. To show multiplicativity, we compare the diagram



exhibiting composition in $\mathrm{End}_{\mathcal{C}/\mathcal{C}_{W_\zeta}}(k)$ with the diagram



exhibiting composition of the corresponding elements in $\mathcal{O}_{\mathcal{H}}(U_\zeta)$.

Commutativity of the diagram (7.2.1) follows immediately from the definition of the map ϕ_W.

□

The commutativity of (7.2.1) immediately implies that the maps ϕ_W of Proposition 7.2 determine a map of presheaves as stated in the following corollary.

Corollary 7.3. *The map* $\phi : \Theta_{\Pi(G)} \to \Psi_G^* \mathcal{O}_{\mathcal{H}}$ *defined by*

$$\phi(U) = \phi_{\pi(G)-U} : \Theta_{\Pi(G)}(U) = \mathrm{End}_{\mathcal{C}/\mathcal{C}_{\Pi(G)-U}}(k) \to \mathcal{O}_{\mathcal{H}}(U)$$

for any open $U \subset \Pi(G)$ *determines a homomorphism of presheaves of* k*-algebras on* $\Pi(G)$.

Proposition 7.4. *Let* $\zeta \in \mathrm{H}^\bullet(G,k)$ *be a homogeneous cohomology class of degree* $n > 0$ *with associated principal closed subset* $W_\zeta \subset \Pi(G)$. *Let* U_ζ *denote* $\Pi(G) - W_\zeta$. *For any* $\alpha \in \mathrm{H}^{nj}(G,k)$, *define*

$$\theta_{W_\zeta}(\alpha/\zeta^j) = (k \stackrel{\zeta^j}{\leftarrow} \Omega^{jn}k \stackrel{\alpha}{\to} k).$$

Then

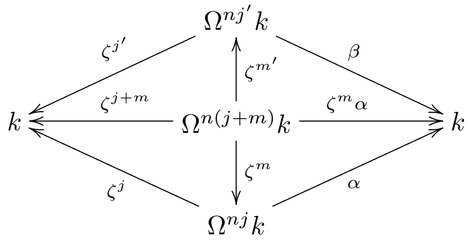
$$\theta_{W_\zeta} : \mathcal{O}_{\mathcal{H}}(U_\zeta) \to \Theta_{\Pi(G)}(U_\zeta)$$

is an isomorphism, inverse to ϕ_{W_ζ}.

In particular, $\Theta_{\Pi(G)}(U_\zeta)$ *is a commutative* k*-algebra.*

Proof. Observe that ζ^j is a $\mathcal{C}/\mathcal{C}_{W_\zeta}$-isomorphism since the kernel of $k \stackrel{\zeta^j}{\leftarrow} \Omega^{jn}k$ is L_{ζ^j} which has support W_ζ. Hence, $\theta_{W_\zeta}(\alpha/\zeta^j) \in \mathrm{End}_{\mathcal{C}/\mathcal{C}_{W_\zeta}}(k)$.

To verify that θ_{W_ζ} is well defined, we must verify that if $\zeta^{m'} \cdot \beta = \zeta^m \cdot \alpha \in \mathrm{H}^{n(m'+j')}(G,k) = \mathrm{H}^{n(m+j)}(G,k)$, then $\theta_{W_\zeta}(\alpha/\zeta^{m+j}) = \theta_{W_\zeta}(\beta/\zeta^{m+j})$. This follows immediately from the equivalence relation describing morphisms in $\mathcal{C}/\mathcal{C}_{W_\zeta}$ together with the existence of the commutative diagram in stmod(G):



It is immediate from the construction that θ_{W_ζ} and ϕ_{W_ζ} are mutually inverse. Thus, both θ_{W_ζ} and ϕ_{W_ζ} are ring isomorphisms. □

Let $\Phi_G : \operatorname{Proj} \mathrm{H}^\bullet(G,k) \to \Pi(G)$ be the inverse to the homeomorphism Ψ_G of Theorem 3.6. By the universal property of the associated sheaf, the map of presheaves $\Theta_{\Pi(G)} \to \mathcal{O}_\mathcal{H}$ of Corollary 7.3 induces a map of sheaves on $\Pi(G)$

$$\Phi_G^\# = \widetilde{\phi} : \widetilde{\Theta}_{\Pi(G)} \to \Psi_G^* \mathcal{O}_\mathcal{H} \cong \Phi_{G*}\mathcal{O}_\mathcal{H}.$$

In some sense, the following theorem is the ultimate generalization and refinement of "Carlson's Conjecture" which proposed the comparison of rank varieties and cohomological support varieties for kE-modules, where k was assumed to be algebraically closed of characteristic p and E an elementary abelian p-group.

Theorem 7.5. *Let G be a finite group scheme over a field k of positive characteristic. There is an isomorphism of ringed spaces*

$$(\Phi_G, \Phi_G^\#) : (\operatorname{Proj} \mathrm{H}^\bullet(G,k), \mathcal{O}_{\operatorname{Proj} \mathrm{H}^\bullet(G,k)}) \xrightarrow{\sim} (\Pi(G), \widetilde{\Theta}_{\Pi(G)})$$

given by the homeomorphism $\Phi_G : \operatorname{Proj} \mathrm{H}^\bullet(G,k) \to \Pi(G)$ and sheaf isomorphism $\Phi_G^\# : \widetilde{\Theta}_{\Pi(G)} \to \Phi_{G*}\mathcal{O}_{\operatorname{Proj} \mathrm{H}^\bullet(G,k)}$.

Proof. We only have to justify that $\widetilde{\phi}$ is an isomorphism of sheaves. As before, let $\mathcal{H} = \operatorname{Proj} \mathrm{H}^\bullet(G,k)$. Let $\zeta \in \mathrm{H}^n(G,k)$ where n is even if $p > 2$, and let $W_\zeta = \Pi(G)_{L_\zeta}$. By Proposition 3.7, $\Phi_G^{-1}(W_\zeta) = V(\zeta)$. Thus,

$$\{U_\zeta; \zeta \in \mathrm{H}^\bullet(G,k)\}, \quad \{\mathcal{H} - V(\zeta); \zeta \in \mathrm{H}^\bullet(G,k)\}$$

give bases for the topologies on $\Pi(G)$ and $\mathcal{H}$ respectively. Since $\phi(U_\zeta) : \Theta_{\Pi(G)}(U_\zeta) \to (\Phi_{G*}\mathcal{O}_\mathcal{H})(U_\zeta) = \mathcal{O}_\mathcal{H}(\Phi_G^{-1}(U_\zeta))$ is an isomorphism for any ζ by Proposition 7.4, we conclude that $\Phi_G^\# = \widetilde{\phi} : \widetilde{\Theta}_{\Pi(G)} \to \Phi_{G*}\mathcal{O}_\mathcal{H}$ induces an isomorphism on stalks and thus is a sheaf isomorphism.

□

Corollary 7.6. *(of the proof.) The presheaf $\Theta_{\Pi(G)}$ and its associated sheaf $\widetilde{\Theta}_{\Pi(G)}$ take the same values on the basic open sets of the form $\Pi(G) - W_\zeta$.*

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