# DETECTING NILPOTENCE AND PROJECTIVITY OVER FINITE UNIPOTENT SUPERGROUP SCHEMES

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ABSTRACT. This work concerns the representation theory and cohomology of a finite unipotent supergroup scheme $G$ over a perfect field $k$ of positive characteristic $p \geq 3$. It is proved that an element $x$ in the cohomology of $G$ is nilpotent if and only if for every extension field $K$ of $k$ and every elementary sub-supergroup scheme $E \subseteq G_K$, the restriction of $x_K$ to $E$ is nilpotent. It is also shown that a $kG$-module $M$ is projective if and only if for every extension field $K$ of $k$ and every elementary sub-supergroup scheme $E \subseteq G_K$, the restriction of $M_K$ to $E$ is projective. The statements are motivated by, and are analogues of, similar results for finite groups and finite group schemes, but the structure of elementary supergroups schemes necessary for detection is more complicated than in either of these cases. One application is a detection theorem for the nilpotence of cohomology, and projectivity of modules, over finite dimensional Hopf subalgebras of the Steenrod algebra.

## CONTENTS

# 1. Introduction

There has been considerable research, some of recent vintage, aimed at understanding representations of finite group schemes through the lens of their support varieties; see [3, 4, 5, 6, 10, 12, 11, 27, 28]. The paradigm for these developments is the work on the modular representation theory of finite groups due to Alperin and Evens [1], Avrunin and Scott [2], Chouinard [18], Carlson [16], Dade [19], Quillen [38], among others. This paper is part of a project aimed at finding analogues of some of these results and techniques for finite supergroup schemes. The first step in this direction was taken by Drupieski [21, 22], who proved finite generation of cohomology for finite supergroup schemes, generalizing the theorem of Friedlander and Suslin for finite group schemes [29]. Drupieski and Kujawa [23, 24, 25] have initiated a study of support varieties for restricted Lie superalgebras.

A starting point for any theory of support varieties is the identification of a family of subgroups that detect nilpotence of cohomology classes and projectivity of representations. Once again, finite groups provide a model: Quillen [38] proved that a class in mod $p$ cohomology of a finite group $G$ is nilpotent if (and only if) its restriction to any elementary abelian $p$-subgroup $E < G$ is nilpotent in $H^*(E, \overline{\mathbb{F}}_p)$; see also Quillen and Venkov [39]. This detection theorem is a key ingredient in the proof of Quillen's stratification theorem that gives a complete description of the Zariski spectrum of $H^*(G, \overline{\mathbb{F}}_p)$. Around the same time, Chouinard [18] proved that a representation $M$ of $G$ is projective if (and only if) the restriction of $M$ to any elementary abelian $p$-subgroups $E < G$ is projective.

In this work we establish analogues of the results of Quillen and Chouinard for finite supergroup schemes. Throughout we fix a perfect field $k$ of positive characteristic $p \geq 3$. A finite supergroup scheme over $k$ may be viewed either as a functor on the category of $\mathbb{Z}/2$-graded commutative $k$-algebras with values in finite groups, or a finite dimensional $\mathbb{Z}/2$-graded cocommutative Hopf algebra; see Section 2 for details. The focus will be on unipotent supergroup schemes, though some of the preliminary results apply more generally. Each finite supergroup scheme has an *even* part which is a finite group scheme. In turn any finite group or group scheme furnishes an example of a supergroup scheme, but there are many more. Notably, the odd version of the additive group $\mathbb{G}_a$, denoted $\mathbb{G}_a^-$ and defined as a functor by $\mathbb{G}_a^-(R) = R_1^+$, the additive group on the odd part of $R$. The corresponding Hopf algebra is $k[\sigma]/(\sigma^2)$, where $\sigma$ is in odd degree and a primitive element.

The notion of an "elementary" supergroup scheme is a lot more involved than in the case of finite groups. To begin with, we construct a two-parameter family of finite supergroup schemes related to the Witt vectors, denoted $E_{m,n}^-$, with $m \geq 2, n \geq 1$; see Construction 8.5. For example, $E_{m,1}^-$ can be realized as an extension of $\mathbb{G}_a^-$ by $W_{m,1}$, the first Frobenius kernel of Witt vectors of length $m$, recalled in the Appendix. Also $E_{1,n}^- \cong \mathbb{G}_{a(n)} \times \mathbb{G}_a^-$, where $\mathbb{G}_{a(n)}$ is the $n$th Frobenius kernel of $\mathbb{G}_a$.

**Definition 1.1.** A finite supergroup scheme $E$ over $k$ is *elementary* if it is isomorphic to a quotient of some $E_{m,n}^- \times (\mathbb{Z}/p)^{\times s}$.

A special role is played by the quotients of $E_{m,n}^-$ by an even subgroup scheme; these are the *Witt elementary* supergroup schemes, and described completely in Theorem 8.13. Besides the $E_{m,n}^-$ themselves, one has also finite supergroup schemes that we denote $E_{m,n,\mu}^-$, involving an element $\mu$ in $k^\times/(k^\times)^{p^{m+n}-1}$. The Hopf algebra corresponding to $E_{m,n,\mu}^-$ is described in 8.10.

2

Any elementary supergroup scheme is of the form $E \cong E^0 \times (\mathbb{Z}/p)^{\times s}$ where $E^0$ is isomorphic to either $\mathbb{G}_{a(r)}$ or a Witt elementary supergroup scheme.

The group algebra $kE$ of an elementary finite supergroup scheme $E$ is isomorphic to a tensor product of algebras of the form

    (i) $k[s]/(s^p)$
    (ii) $k[\sigma]/(\sigma^2)$, and
    (iii) $k[s,\sigma]/(s^{p^n}, \sigma^2 - s^p)$, where $n \geq 1$,

with $|s|$ even and $|\sigma|$ odd. In particular, as an ungraded algebra $kE$ is a commutative complete intersection, even though case (iii) is not graded commutative.

Our main detection theorem is proved in Section 11.

**Theorem 1.2.** *Let $G$ be a finite unipotent supergroup scheme over a field $k$ of positive characteristic $p \geq 3$. Then the following hold.*

    (i) *An element $x \in H^{*,*}(G, k)$ is nilpotent if and only if for every extension field $K$ of $k$ and every elementary sub-supergroup scheme $E$ of $G_K$, the restriction of $x_K \in H^{*,*}(G_K, K)$ to $H^{*,*}(E, K)$ is nilpotent.*

    (ii) *A $kG$-module $M$ is projective if and only if for every extension field $K$ of $k$ and every elementary sub-supergroup scheme $E$ of $G_K$, the restriction of $M_K$ to $E$ is projective.*

We also prove two versions of (i) for arbitrary coefficients. Theorem 11.1(i) proves the detection of nilpotents for $H^{*,*}(G, M)$ for any $G$-module $M$ where nilpotents are understood in the sense of Definition 6.1. Theorem 11.2, which generalizes a theorem of Bendel [3] for unipotent group schemes, gives detection of nilpotents for $H^{*,*}(G, \Lambda)$ with coefficients in a unital $G$-algebra $\Lambda$.

We also formulate and prove $\mathbb{Z}$-graded versions of our theorems, and apply them to finite dimensional subalgebras of the Steenrod algebra over $\mathbb{F}_p$. The structure of the Steenrod algebra is well understood and the detection theorem in that case takes on a particularly simple form; see Theorem 12.4.

**Looking ahead.** Our results only cover unipotent supergroup schemes, and it would be interesting to understand what more needs to be done in order to cover the general case. Unlike the case of finite group schemes, for a general finite supergroup scheme it is not true that cohomology modulo nilpotents and projectivity of modules are detected on unipotent sub-supergroup schemes. Conversations with Chris Drupieski lead us to suspect that there is a mild generalisation of the Witt elementaries that are not unipotent, but which leads to a suitable detection family in this context.

In a different direction, the detection theorems are only the first steps towards developing a theory of support varieties. Again we turn to groups to show us the way: While Chouinard's work highlights the role of elementary abelian groups, Dade [19] proved that to detect projectivity of a representation of an elementary abelian $p$-group $E$, one can restrict further to all *cyclic shifted subgroups* of the group algebra $kE$, which then becomes purely a problem in linear algebra. This detection theorem, now known as "Dade's lemma", is the foundation for the theory of rank varieties for modules for finite groups pioneered by Carlson [16], and further developed by Benson, Carlson, and Rickard [9]. Their work was absorbed and generalised to the theory of $\pi$-points for finite groups schemes by Friedlander and Pevtsova [27, 28].

Theorem 1.2 opens up the road to a theory of $\pi$-points for finite unipotent supergroup schemes. We take this up in a follow up paper [13], where it is used to establish a stratification theorem for the stable module category, akin to the one in [11].

**Structure of the paper.** The strategy of the proof of Theorem 1.2 is quite intricate and we found it expedient to divide the paper into two parts. Before delving into a summary of the parts, it would perhaps help to present a roadmap of the proof; it follows the one for finite unipotent group schemes given in [3], but a number of extra complications arise.

The simplest scenario is that there is a surjective map from $G$ to either $\mathbb{G}_a^- \times \mathbb{G}_a^-$ or $\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$, for then the argument in [3, Theorem 8.1] applies. Otherwise one reduces to the case where there is a surjective map $f\colon G \to \mathbb{G}_{a(r)} \times (\mathbb{G}_a^-)^\epsilon \times (\mathbb{Z}/p)^s$ with $r, s \geq 0$ and $\epsilon = 0$ or 1, such that $H^{1,*}(f)$ is an isomorphism. It is easy to tackle the case when $f$ is itself an isomorphism. When it is not an isomorphism, a standard argument yields that $H^{2,*}(f)$ has a kernel. The situation when this kernel contains an element of odd degree, that is to say, when $H^{2,1}(f)$ is not one-to-one, is dealt with in [14]. The difficulty arises when the kernel of $H^{2,*}(f)$ is concentrated in even degrees. Even here there are two cases, as elaborated on further below. The first one allows us to drop to proper subgroups and is easy to handle. The second one leads to elementary supergroup schemes. This is where the major deviation from [3] occurs, and requires the bulk of the work. It occupies Part II of this paper.

Here is a more detailed description of the paper: Part I, comprising Sections 2 to 7, provides background material on finite supergroup schemes and extensions of a number of techniques used in other contexts. Section 2 starts things off with main definitions, examples, and basic properties of supergroup schemes. Section 3 records some key facts on low degree cohomology modules. Section 4 describes the action of Steenrod operations on the cohomology of finite supergroup schemes. The central calculation there is Theorem 4.3 that establishes that a homogeneous ideal in $H^{*,*}(\mathbb{G}_{a(r)} \times \mathbb{G}_a^- \times (\mathbb{Z}/p)^{\times s}, k)$ stable under the Steenrod operations and containing an element from $H^{2,0}$ must have an element of a specific form. The proof follows closely the proofs of the analogous result for $(\mathbb{Z}/p)^{\times s}$, due to Serre [40], for $\mathbb{G}_{a(r)}$, due to Bendel, Friedlander, and Suslin [6], and for $\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^{\times s}$ due to Bendel [3], but the conclusion is different. Whereas for finite group schemes, such an ideal always has an element that is a product of Bocksteins of elements in degree 1, in the super case we get either a product of Bocksteins or a mysterious element $\zeta^2 - \gamma x_r$ with $|\zeta| = (1,1)$, $|x_r| = (2,0)$, $\gamma \in k$. This element is responsible for the work we have to do in Part II.

Part I culminates in Theorem 7.2 that asserts that if a finite unipotent supergroup satisfies certain conditions, laid out in Hypothesis 7.1, nilpotence (of cohomology elements) and projecitivty (of modules) are detected on proper sub-supergroup schemes after field extensions. For finite group schemes (not super ones) the calculation with the Steenrod operations in Section 4 would then yield that any unipotent group scheme that is not isomorphic to $\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^{\times s}$ satisfies Hypothesis 7.1. And this is precisely the argument in Bendel [3]. Thus, up to the end of Part I we are mostly mimicking the techniques existing in the literature. Life in the super world turns out to be more complicated, all because of the cohomology class $\zeta^2 - \gamma x_r$ that cannot be eliminated with the help of the Steenrod operations. The task of the second part of the paper is to show that if a finite unipotent supergroup scheme does not satisfy Hypothesis 7.1, then, in fact, it must be elementary.

Part II begins in Section 8 with the construction of the elementary supergroup schemes featuring in the statement of Theorem 1.2. Their cohomology rings are calculated in Section

9. These calculations feed into the proof of Theorem 10.3 that is a cohomological criterion for recognising elementary supergroup schemes. Theorem 1.2 is proved as Theorem 11.1. Its consequences for the Steenrod algebra are described in Section 12. The Appendix provides background on Dieudonné modules needed to describe elementary supergroup schemes.

## Part 1. **Recollections**

## 2. Finite supergroup schemes

We give a compressed introduction to the terminology we shall employ in the paper referring the reader to a number of excellent sources on super vector spaces, super algebras and super groups schemes, such as, for example, a survey paper by A. Masuoka [34] or [23].

Throughout this manuscript $k$ will be a field of positive characteristic $p \geq 3$. We assume $k$ is perfect since some of the structural results for supergroup schemes require that condition. It is clear that the main theorem holds for an arbitrary field $k$ of characteristic $p$ once it is proved for a perfect field of the same characteristic.

An *affine supergroup scheme* over $k$ is a covariant functor from $\mathbb{Z}/2$-graded commutative $k$-algebras (in the sense that $yx = (-1)^{|x||y|}xy$) to groups, whose underlying functor to sets is representable. If $G$ is a supergroup scheme then its *coordinate ring $k[G]$* is the representing object. By applying Yoneda's lemma to the group multiplication and inverse maps, it is a $\mathbb{Z}/2$-graded commutative Hopf algebra. We denote the comultiplication on $k[G]$ by $\Delta \colon k[G] \to k[G] \otimes k[G]$ and the counit map by $\epsilon \colon k[G] \to k$ with $I = \ker \epsilon$ being the augmentation ideal and note that these are *degree-preserving* (equivalently, even) algebra homomorphisms. The correspondence between affine supergroup schemes and their coordinate algebras gives a contravariant equivalence of categories between affine supergroup schemes and $\mathbb{Z}/2$-graded commutative Hopf algebras.

A *finite* supergroup scheme $G$ is an affine supergroup scheme whose coordinate ring is finite dimensional. In this case, the dual $kG = \mathrm{Hom}_k(k[G], k)$ is a finite dimensional $\mathbb{Z}/2$-graded cocommutative Hopf algebra called the *group ring* of $G$. This gives a covariant equivalence of categories between finite supergroup schemes and finite dimensional $\mathbb{Z}/2$-graded (equivalently, "super") cocommutative Hopf algebras.

$$
(2.1) \qquad \left\{ \begin{array}{c} \text{finite super-} \\ \text{group schemes} \end{array} \right\} \quad \sim \quad \left\{ \begin{array}{c} \text{finite dimensional super-} \\ \text{cocommutative Hopf algebras} \end{array} \right\}
$$

We employ the notation $V = V_0 \oplus V_1$ for $\mathbb{Z}/2$-graded (equivalently, "super") vector spaces, where $V_0$ are the *even* degree elements, and $V_1$ are the *odd* degree elements. *A $kG$-module* is a $\mathbb{Z}/2$-graded $k$-vector space on which $kG$ acts respecting the grading in the usual way. As in the ungraded setting, a $kG$-module has an equivalent description as a rational representation of the supergroup $G$ on the category of super vector spaces. We consider all modules including infinite dimensional ones. The trivial module $k$ is the trivial one dimensional representation concentrated in the even degree.

If $K$ is a field extension of $k$, and $G$ is an affine supergroup scheme, we write $K[G]$ for $K \otimes_k k[G]$, which is a graded commutative Hopf algebra over $K$. This defines a supergroup scheme over $K$ denoted $G_K$, and we have a natural isomorphism of Hopf superalgebras $KG_K \cong K \otimes_k kG$.

For each $kG$-module $M$, we set

$$M_K := K \otimes_k M \quad \text{and} \quad M^K := \operatorname{Hom}_k(K, M),$$

viewed as $KG_K$-modules.

The *even part* $G_{\mathsf{ev}}$ of an affine supergroup scheme $G$ is the largest sub-supergroup scheme whose coordinate ring contains no odd degree elements (see [34]). It may be regarded as an affine group scheme. Its coordinate ring $k[G_{\mathsf{ev}}]$ is the quotient of $k[G]$ by the ideal generated by the odd degree elements. This ideal is automatically a Hopf ideal, since the coproduct $\Delta$ applied to an odd degree element is necessarily a linear combination of tensors $a \otimes b$ where either $a$ or $b$ is odd. An *even subgroup scheme* of $G$ is a subgroup scheme of $G_{\mathsf{ev}}$.

**Example 2.1.** Any affine group scheme $G$ may be thought of as an affine supergroup scheme with $G = G_{\mathsf{ev}}$.

Another way to look at the assignment $G \mapsto G_{\mathsf{ev}}$ is that it gives the right adjoint to the inclusion functor from the affine group schemes to affine supergroups schemes.

**Definition 2.2.** If $G$ is an affine supergroup scheme, let $G^{(1)}$ be the base change of $G$ via the Frobenius map $x \mapsto x^p$ on $k$. Then the *Frobenius map* $F \colon G \to G^{(1)}$ corresponds to the map of coordinate rings $k[G^{(1)}] \to k[G]$ given by $x \to x^p$. The $r$th *Frobenius kernel* $G_{(r)}$ of $G$ is defined to be the kernel of the iterate $F^r \colon G \to G^{(r)}$.

**Convention 2.3.** By $G_{(0)}$ we always mean the trivial group scheme.

**Definition 2.4.** A finite supergroup scheme $G$ over $k$ is said to be *unipotent* if $k$ is the unique irreducible $kG$-module, which may be either in even or odd degree. A supergroup scheme $G$ is *connected* if $k[G]$ is local.

If $G$ is a finite connected supergroup scheme then for some $r \geq 0$ we have $G = G_{(r)}$. The least such value of $r$ is called the *height* of $G$. Note that $G$ has height zero if and only if $G$ is the trivial supergroup scheme.

**Lemma 2.5.** *Any finite supergroup scheme $G$ is a semidirect product $G^0 \rtimes \pi_0(G)$ with $G^0$ connected and $\pi_0(G)$ the finite group of connected components.*

*Proof.* See Lemma 5.3.1 of Drupieski [21]. The proof uses the fact that $k$ is perfect and has odd prime characteristic. $\square$

**Theorem 2.6.** *Let $G$ be a connected finite supergroup scheme. Then there exist odd degree elements $y_1, \ldots, y_n \in k[G]$ such that we have an isomorphism of $\mathbb{Z}/2$-graded $k$-algebras*

$$k[G] \cong k[G_{\mathsf{ev}}] \otimes \Lambda(y_1, \ldots, y_n).$$

*In particular, if $G_{\text{ev}}$ is non-trivial then $G$ has the same height as $G_{\text{ev}}$.*

*Proof.* Let $I$ be the augmentation ideal of $k[G]$. Pick odd elements $\{y_1, \ldots, y_n\}$ such that their residues give a basis of the odd part of the super vector space $I/I^2$. Then the ideal $(y_1, \ldots, y_n)$ is a Hopf ideal, and we have an isomorphism $k[G]/(y_1, \ldots, y_n) \cong k[G_{\text{ev}}]$. Since $k[G_{\text{ev}}]$ is a connected finite group scheme, we can find algebraic generators $x'_1, \ldots, x'_m \in k[G_{\text{ev}}]$ such that $k[G_{\text{ev}}]$ is a truncated polynomial algebra on these generators ([43, 14.4]). Let $x_1, \ldots, x_m \in I$ be even liftings of $x'_1, \ldots, x'_m$ to $k[G]$, and let $B$ be the (even) subalgebra of $k[G]$ generated by $x_1, \ldots, x_m$. By construction $\{x_1, \ldots, x_m\}$ give a basis of the even part of $I/I^2$. Moreover, the odd elements $y_1, \ldots, y_n$ square to zero and (super) commute, hence, generate a copy of $\Lambda(y_1, \ldots, y_n)$ in $k[G]$. We therefore have a surjective map

$$f \colon B \otimes \Lambda(y_1, \ldots, y_n) \to k[G]$$

We wish to show that this map is an isomorphism of algebras. The augmentation ideals are the same on both sides by constructions, and, hence, it suffices to show that $f$ induces an isomorphism on the associated graded algebras. Note that $\operatorname{gr} k[G] \cong \bigoplus I^i/I^{i+1}$ inherits the structure of a Hopf algebra.

If $f$ is not an isomorphism, its kernel contains a nonzero polynomial involving both $x_i$ and $y_i$. Choose one which involves the minimal number of the variables $y_i$, let $r$ be the maximal index such that this polynomial involves $y_r$, and write it in the form

$$a + by_r = 0$$

where $a$ and $b$ only involve $B$ and $y_1, \ldots, y_{r-1}$. Apply the coproduct map $\Delta$ to obtain

$$\Delta(a) + \Delta(b)(y_r \otimes 1 + 1 \otimes y_r) = 0 \,.$$

Since $\Delta(b) = b \otimes 1 + 1 \otimes b + I \otimes I$ ([30, I.2]), there is a term $b \otimes y_r$ in the sum which must vanish. We conclude that $b = 0$ and, hence, $a = 0$, contradicting the minimality of $r$. This proves that $f$ is an isomorphism. In particular, $B$ does not intersect the ideal $(y_1, \ldots, y_n)$, and so the projection map $k[G] \to k[G]/(y_1, \ldots, y_n) \cong k[G_{\text{ev}}]$ induces an isomorphism $B \cong k[G_{\text{ev}}]$. $\square$

*Remark* 2.7.    (1) Masuoka [33, Theorem 4.5] proves, without the finiteness hypothesis, that there is counital algebra isomorphism $k[G] \cong k[G_{\text{ev}}] \otimes \Lambda((\operatorname{Lie} G)^{\#}_{\text{odd}})$.
   (2) Since $k[\pi_0(G)]$ sits in even degree, Lemma 2.5 implies that the tensor decomposition of Theorem 2.6 holds for any finite supergroup scheme.
   (3) The structure of the coordinate ring of an ungraded finite connected group scheme is known ([43, Theorem 14.4]). Putting it together with Theorem 2.6, we conclude that for any finite connected supergroup scheme $G$ there exists a $k$-algebra isomorphism

$$k[G] \cong k[x_1, \ldots, x_n]/(x_1^{p^{i_1}}, \ldots, x_n^{p^{i_n}}) \otimes \Lambda(y_1, \ldots, y_m)$$

      where $x_i$ are even and $y_j$ are odd.
   (4) The Frobenius map $F \colon k[G^{(1)}] \to k[G]$ kills $k[G^{(1)}]_{\text{odd}}$ since odd elements square to 0 by supercommutativity. Hence, the image of $F$ lands in $k[G_{\text{ev}}]$, that is, the composite $k[G/G_{(1)}] \to k[G] \to k[G_{\text{ev}}]$ is injective.

**Corollary 2.8.** *If $G$ is a finite supergroup scheme then $G = G_{\text{ev}} G_{(1)}$.*

*Proof.* It follows from Lemma 2.5 that $G_{\text{ev}} = G_{\text{ev}}^0 \rtimes \pi_0(G)$. So we may assume that $G$ is connected. It then follows from Theorem 2.6 (see By Remark 2.7 the composite $k[G/G_{(1)}] \to$

$k[G] \to k[G_{\mathrm{ev}}]$ is injective. Since this is an injective map of Hopf algebras, is it faithfully flat (see, for example, [43, Theorem 14.1]) and, therefore, the corresponding map on group schemes $G_{\mathrm{ev}} \to G \to G/G_{(1)}$ is surjective. Hence, $G = G_{\mathrm{ev}}G_{(1)}$. $\square$

**Warning 2.9.** The subgroup $G_{(1)}$ is normal in $G$, but $G_{\mathrm{ev}}$ need not be normal.

**Example 2.10.** The additive (super)group scheme $\mathbb{G}_a$ is a purely even group scheme, given by the assignment
$$\mathbb{G}_a(R) = R_0^+,$$
where $R_0^+$ the additive group on the even part of a superalgebra $R$. We have $k[\mathbb{G}_a] = k[T]$ with the $T$ primitive in even degree. The Frobenius kernels $\mathbb{G}_{a(r)}$ are purely even connected unipotent supergroup schemes with $k[\mathbb{G}_{a(r)}] = k[T]/T^{p^r}$, and $T$ primitive.

**Example 2.11.** We denote by $\mathbb{G}_a^-$ the finite supergroup scheme such that $k\mathbb{G}_a^- = k[\sigma]/(\sigma^2)$ with $\sigma$ primitive in odd degree. Then $\mathbb{G}_a^-$ is connected and unipotent. As a functor, $\mathbb{G}_a^-$ is defined by $\mathbb{G}_a^-(R) = R_1^+$, the additive group on the odd part of a superalgebra $R$.

More generally, let $V$ be a finite-dimensional vector space, and let $\Lambda^*(V)$ be the $\mathbb{Z}/2$-graded exterior algebra on $V$ where the elements of $V$ are primitive of odd degree. With this convention, $\Lambda^*(V)$ becomes a supercommutative Hopf algebra and, hence, is isomorphic to a group algebra of a product of copies of $\mathbb{G}_a^-$, and hence corresponds to a connected unipotent finite supergroup scheme.

**Example 2.12.** Let $W_{1,1}^-$ be the finite supergroup scheme such that $kW_{1,1}^- = k[\sigma]/(\sigma^{2p})$ with $\sigma$ primitive in odd degree. Then $W_{1,1}^-$ has height 1 and sits in a nonsplit short exact sequence

$$(2.2) \qquad\qquad 1 \to \mathbb{G}_{a(1)} \to W_{1,1}^- \to \mathbb{G}_a^- \to 1.$$

More generally, let $W_{m,1}^-$ be the finite supergroup scheme with $kW_{m,1}^- = k[\sigma]/(\sigma^{2p^m})$ where $\sigma$ is primitive in odd degree and $m \geq 1$. Then $W_{m,1}^-$ has height one, and it sits in a nonsplit short exact sequence

$$(2.3) \qquad\qquad 1 \to W_{m,1} \to W_{m,1}^- \to \mathbb{G}_a^- \to 1$$

where $W_{m,1}$ denotes the Witt vectors of length $m$ and height one as described in Appendix A, whose group algebra is $kW_{m,1} = k[s]/(s^{p^m})$, and $s = \sigma^2$ is primitive in even degree.

**Example 2.13.** A *p-restricted Lie superalgebra* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}/2$-graded Lie algebra with a $p$-restriction map on the even part, and such that the odd part is a $p$-restricted module over the even part. The $p$-restricted enveloping algebra $\mathcal{U}^{[p]}(\mathfrak{g})$ is the group algebra of a connected finite supergroup scheme which is unipotent if and only if $\mathfrak{g}$ is unipotent.

**Lemma 2.14.** *Let $G$ be a finite supergroup scheme. Then the primitive elements in $kG$ form a p-restricted Lie superalgebra $\mathfrak{g} = \mathsf{Lie}(G)$ over $k$ with Lie bracket given by commutator and p-restriction map given by the p-power map in $kG$. The natural map $\mathcal{U}^{[p]}(\mathfrak{g}) \to kG$ induces an isomorphism $\mathcal{U}^{[p]}(\mathfrak{g}) \to kG_{(1)}$.*

*Proof.* See Lemma 4.4.2 of Drupieski [21]. $\square$

**Example 2.15.** Example 2.12 has height one, so is of the form $\mathcal{U}^{[p]}(\mathfrak{g})$. The $p$-restricted Lie superalgebra $\mathfrak{g}$ is generated by an element $\sigma$ in odd degree with relation $[\sigma, \sigma]^{[p^m]} = 0$.

*Remark* 2.16. If $G$ is a finite connected supergroup scheme of height 1 with the corresponding Lie algebra $\mathfrak{g}$, then $\mathfrak{g}_0$ is an even (restricted) Lie algebra corresponding to $G_{\mathsf{ev}}$, that is, $\mathcal{U}^{[p]}(\mathfrak{g}_0) \cong kG_{\mathsf{ev}}$.

**Lemma 2.17.** *A finite unipotent supergroup scheme $G$ with $G_{\mathsf{ev}}^0 = 1$ is isomorphic to $(\mathbb{G}_a^-)^{\times r} \rtimes (\mathbb{Z}/p)^{\times s}$.*

*Proof.* The assumption $G_{\mathsf{ev}}^0 = 1$ implies that $G_0$ has height 1, and, hence, corresponds to a Lie superalgebra $\mathfrak{g}$. By Remark 2.16, $\mathfrak{g}_0 = 0$, therefore, $\mathcal{U}^{[p]}(\mathfrak{g}) = \Lambda^*(\mathfrak{g}_1)$, and, hence, $G_0 = (\mathbb{G}_a^-)^{\times \dim \mathfrak{g}_1}$. The statement follows from Lemma 2.5. $\qquad\square$

For sub-supergroup schemes $H, H' \leqslant G$, the commutator sub-supergroup scheme is defined as in [20, II.5.4.8] as a representable functor. We need an analogue of the following standard result in group theory.

**Lemma 2.18.** *Let $G$ be a finite supergroup scheme, and $H, H' \trianglelefteq G$ be normal sub-supergroup schemes. Then $[H, H']$ is normal in $G$.*

*Proof.* It suffices to check pointwise that for $a \in H(R), b \in H'(R)$ and $c \in G(R)$, we have that $[a,b]^c \in [H(R), H'(R)]$, where the latter commutator is as of discrete groups. This follows from the obvious identity

$$c(aba^{-1}b^{-1})c^{-1} = cac^{-1}cbc^{-1}ca^{-1}c^{-1}cb^{-1}c^{-1}. \qquad\square$$

## 3. Low degree cohomology

The cohomology $H^{*,*}(G, k)$ of a finite supergroup scheme $G$ is isomorphic to $\mathrm{Ext}_{kG}^{*,*}(k, k)$. The first index is homological, and the second is the internal $\mathbb{Z}/2$-grading. Drupieski [21, 22] has proved that $H^{*,*}(G, k)$ is a finitely generated $k$-algebra, which is graded commutative in the sense that if $x \in H^{m,\alpha}(G, k)$ and $y \in H^{n,\beta}(G, k)$ then

$$yx = (-1)^{mn}(-1)^{\alpha\beta}xy.$$

We start by identifying the first cohomology group of $G$.

**Lemma 3.1.** *Let $G$ be a finite supergroup scheme with the group of connected components $\pi$. Then we have $H^{1,0}(G, k) = \mathrm{Hom}_{\mathsf{Gr}/k}(G, \mathbb{G}_a)$ and $H^{1,1}(G, k) = \mathrm{Hom}_{\mathsf{sGr}/k}(G^0, \mathbb{G}_a^-)^\pi$. Moreover, $\mathrm{Hom}_{\mathsf{sGr}/k}(G, \mathbb{G}_a) \cong \mathrm{Hom}_{\mathsf{sGr}/k}(G^0, \mathbb{G}_a)^\pi \times \mathrm{Hom}_{\mathsf{Gr}/k}(\pi, \mathbb{G}_a)$.*

*Proof.* Identification of $H^{1,*}$ with Hom follows from the standard cobar resolution used to compute cohomology $H^{*,*}(G, k)$. The last statement is proved as in [3, Lemma 5.1]. $\qquad\square$

**Lemma 3.2.** *If $G$ is a non-trivial unipotent finite supergroup scheme then there is a non-trivial homomorphism from $G$ to either $\mathbb{G}_{a(1)}$ or $\mathbb{G}_a^-$ or $\mathbb{Z}/p$.*

*Proof.* Since $G$ is unipotent, the group of connected components $\pi$ is a $p$-group. If there are no non-trivial maps to $\mathbb{Z}/p$, then $\pi$ is trivial and $G$ is connected. For a finite connected supergroup scheme, if there are no non-trivial homomorphisms from $G$ to $\mathbb{G}_{a(1)}$ then there are also none to $\mathbb{G}_a$. So if there are also no non-trivial homomorphisms from $G$ to $\mathbb{G}_a^-$, Lemma 3.1 yields $\mathrm{Ext}_{kG}^{1,*}(k, k) = 0$. As $kG$ is a local ring this implies $G$ is trivial. $\qquad\square$

9

If $f\colon G \to G'$ is a group homomorphism then $f^*\colon H^{*,*}(G',k) \to H^{*,*}(G,k)$ preserves both the homological and the internal degree, and commutes with the Steenrod operations (to be discussed in Section 4). If $N$ is a normal sub-supergroup scheme of $G$ then there is the Lyndon-Hochschild–Serre spectral sequence

$$H^{*,*}(G/N, H^{*,*}(N,k)) \Rightarrow H^{*,*}(G,k)$$

in which the internal degrees are carried along, and preserved by all the differentials. The spectral sequence also gives the five-term exact sequence:

$$1 \longrightarrow H^1(G/N,k) \longrightarrow H^1(G,k) \longrightarrow H^1(N,k)^{G/N} \xrightarrow{\ d_2\ } H^2(G/N,k) \longrightarrow H^2(G,k).$$

**Lemma 3.3.** *Let $\mathcal{L}' \subset \mathcal{L}$ be Lie superalgebras such that $\mathcal{L}'$ is odd and central in $\mathcal{L}$ and $\mathcal{L}/\mathcal{L}'$ is even. Then $\mathcal{L} \simeq \mathcal{L}' \times \mathcal{L}/\mathcal{L}'$.*

*Proof.* Let $\mathcal{L}_{\mathrm{ev}}$ be the $p$-restricted Lie sub-superalgebra of even elements in $\mathcal{L}$. The assumption implies that it is normal and isomorphic to $\mathcal{L}/\mathcal{L}'$; hence, $\mathcal{L} \simeq \mathcal{L}' \times \mathcal{L}_{\mathrm{ev}}$. $\qquad\square$

**Lemma 3.4.** *If a unipotent finite supergroup scheme $G$ has $H^{1,1}(G,k) = 0$, then $G = G_{\mathrm{ev}}$.*

*Proof.* Since $G/G_{(1)}$ is even by Corollary 2.8, the Lyndon-Hochschild-Serre spectral sequence applied to the supergroup extension $1 \to G_{(1)} \to G \to G/G_{(1)} \to 1$ implies that $H^{*,1}(G,k) = H^{*,1}(G_{(1)},k)$. Hence, the assumption together with Lemma 3.1 imply that there are no non-trivial maps from $G_{(1)}$ to $\mathbb{G}_a^-$. We need to show that $G_{(1)}$ is purely even.

Let $\mathcal{L}$ be the unipotent Lie superalgebra associated with $G_{(1)}$. Since $\mathcal{L}$ is unipotent, we can choose a central series

$$\mathcal{L}_1 \subset \mathcal{L}_0 \subset \mathcal{L}$$

such that $\mathcal{L}/\mathcal{L}_0$ is purely even and $\mathcal{L}_0/\mathcal{L}_1 \simeq \mathsf{Lie}\,\mathbb{G}_a^-$. By Lemma 3.3, we get that $\mathcal{L}/\mathcal{L}_1$ has $\mathsf{Lie}\,\mathbb{G}_a^-$ as a direct factor, so there is a surjective map from $G_{(1)}$ to $\mathbb{G}_a^-$, a contradiction. $\quad\square$

The five term sequence can be used in exactly the same way as in the proof of [5, Lemma 1.2], to prove the following analogue.

**Lemma 3.5.** *Let $f\colon G \to \overline{G}$ be a surjective homomorphism of unipotent supergroup schemes. If $f^*\colon H^{1,*}(\overline{G},k) \to H^{1,*}(G,k)$ is an isomorphism and $f^*\colon H^{2,*}(\overline{G},k) \to H^{2,*}(G,k)$ is injective then $f$ is an isomorphism.* $\qquad\square$

*Remark* 3.6. Lemma 3.1 implies that the condition that $f^*\colon H^{1,*}(\overline{G},k) \to H^{1,*}(G,k)$ is an isomorphism guarantees that any homomorphism from $G$ to $\mathbb{G}_{a(1)}$, $\mathbb{G}_a^-$ and $\mathbb{Z}/p$ factors through $\overline{G}$.

## 4. Steenrod operations

The Steenrod algebra acts on the cohomology of any $\mathbb{Z}$-graded cocommutative Hopf algebra, and hence also on the cohomology of any affine supergroup scheme ([35, Theorem 11.8], [44]). We recall how the Steenrod operations act using the re-indexing introduced in [14]. In order to make the indexing work for $\mathbb{Z}/2$-graded algebra, we index with half-integers.

For $p$ odd, there are natural operations

$$\mathscr{P}^i\colon H^{s,t}(G,k) \to H^{s+2i(p-1),pt}(G,k)$$

$$\beta\mathscr{P}^i\colon H^{s,t}(G,k) \to H^{s+1+2i(p-1),pt}(G,k),$$

defined in the following cases: when $t$ is even, then $i \in \mathbb{Z}$, and if $t$ is odd, then $i \in \mathbb{Z} + \frac{1}{2}$. Note that since $p$ is odd, $pt$ is congruent to $t$ mod 2, so the operations preserve internal degree as elements of $\mathbb{Z}/2$.

The Steenrod operations satisfy the following properties:

(i) $\mathscr{P}^i = 0$ if either $i < 0$ or $i > s/2$,
$\beta \mathscr{P}^i = 0$ if either $i < 0$ or $i \geq s/2$;
(ii) Semi-linearity: $\mathscr{P}^i(ax) = a^p \mathscr{P}^i(x)$ for $a \in k$;
(iii) $\mathscr{P}^i(x) = x^p$ if $i = s/2$;
(iv) Cartan formula:
$\mathscr{P}^j(xy) = \sum_i \mathscr{P}^i(x)\mathscr{P}^{j-i}(y)$,
$\beta \mathscr{P}^j(xy) = \sum_i (\beta \mathscr{P}^i(x)\mathscr{P}^{j-i}(y) + \mathscr{P}^i(x)\beta\mathscr{P}^{j-i}(y))$;
(v) The $\mathscr{P}^i$ and $\beta \mathscr{P}^i$ satisfy the Adem relations.

We record its action on $H^{*,*}(\mathbb{G}_a^-, k)$ (see [14, Proposition 3.1]).

**Proposition 4.1.** *One has $H^{*,*}(\mathbb{G}_a^-, k) \cong k[\zeta]$, a polynomial ring on $\zeta$ in degree $(1,1)$. The action of the Steenrod operations on $H^{*,*}(\mathbb{G}_a^-, k)$ are given by $\mathscr{P}^{\frac{1}{2}}(\zeta) = \zeta^p$, $\beta \mathscr{P}^{\frac{1}{2}}(\zeta) = 0$.* $\square$

Next, we describe the analogue of Proposition 3.6 of [3] for $G = \mathbb{G}_{a(r)} \times (\mathbb{G}_a^-)^\varepsilon \times (\mathbb{Z}/p)^{\times s}$ with $r, s \geq 0$, $\varepsilon = 0$ or 1. If $\varepsilon = 1$ we have

$$H^{*,*}(G, k) = k[x_1, \ldots, x_r] \otimes \Lambda(\lambda_1, \ldots, \lambda_r) \otimes k[\zeta] \otimes k[z_1, \ldots, z_s] \otimes \Lambda(y_1, \ldots, y_s)$$

while if $\varepsilon = 0$ the term $k[\zeta]$ is missing. The degrees and action of the Steenrod algebra are as follows. We make the assumption that $\lambda_{r+1} = 0 = x_{r+1}$, that is, $\mathscr{P}^0$ kills $\lambda_r$ and $x_r$.

| | degree | $\mathscr{P}^0$ | $\beta\mathscr{P}^0$ | $\mathscr{P}^{\frac{1}{2}}$ | $\beta\mathscr{P}^{\frac{1}{2}}$ | $\mathscr{P}^1$ | $\mathscr{P}^i$ $(i \geq 2)$ | $\beta\mathscr{P}^i$ $(i \geq 1)$ |
|---|---|---|---|---|---|---|---|---|
| $\lambda_i$ | $(1,0)$ | $\lambda_{i+1}$ | $-x_i$ | | | $0$ | $0$ | $0$ |
| $y_i$ | $(1,0)$ | $y_i$ | $z_i$ | | | $0$ | $0$ | $0$ |
| $\zeta$ | $(1,1)$ | | | $\zeta^p$ | $0$ | | | |
| $x_i$ | $(2,0)$ | $x_{i+1}$ | $0$ | | | $x_i^p$ | $0$ | $0$ |
| $z_i$ | $(2,0)$ | $z_i$ | $0$ | | | $z_i^p$ | $0$ | $0$ |

TABLE 1. Steenrod operations

We recall the following theorem of Serre [40] which is a prototype for both Proposition 3.6 of [3] and Theorem 4.3 and will be used in the proof. The precise result we quote is a special case of Proposition 3.2 of [42]; it differs slightly from Serre's original formulation since we need to consider arbitrary coefficients, not just $\mathbb{F}_p$.

**Theorem 4.2.** *Let $I$ be a homogeneous ideal in $H^*((\mathbb{Z}/p)^{\times s}, k)$ stable under the Steenrod operations. If $I$ contains a nonzero element of degree two, then there exists a finite family $\{u_i\} \subset H^2((\mathbb{Z}/p)^{\times s}, k)$, each of which is a non-trivial linear combination of $\{z_j\}$ with coefficients in $\mathbb{F}_p$ such that the product $\prod u_i \in H^*((\mathbb{Z}/p)^{\times s}, k)$ lies in $I$.* $\square$

Let $f: G \to \overline{G} \cong \mathbb{G}_{a(r)} \times (\mathbb{G}_a^-)^\epsilon \times (\mathbb{Z}/p)^{\times s}$, $(r, s \geq 0, \epsilon = 0$ or 1$)$ be a surjective map of finite unipotent supergroup schemes. The proof of Theorem 11.1 uses in an essential way the description of the kernel of the induced map on degree 2 cohomology

(4.1) $$f^*: H^{2,*}(\overline{G}, k) \to H^{2,*}(G, k)$$

*under the assumption* that

$$f^* \colon H^{1,*}(\overline{G}, k) \xrightarrow{\sim} H^{1,*}(G, k)$$

is an isomorphism. There are two scenarios: Theorem 4.3 deals with the case when the kernel $I = \ker f^*$ has an element of degree $(2, 0)$ whereas Theorem 4.4 considers the case of degree $(2, 1)$. We use extensively the observation that $I$ is stable under the Steenrod operations.

The following theorem includes the case $\overline{G} = (\mathbb{G}_a^-)^\varepsilon \times (\mathbb{Z}/p)^{\times s}$ in disguise; it corresponds to $r = 0$ per our convention that $\mathbb{G}_{a(0)} = 1$.

**Theorem 4.3.** *Let $\overline{G} = \mathbb{G}_{a(r)} \times (\mathbb{G}_a^-)^\varepsilon \times (\mathbb{Z}/p)^{\times s}$, with $r, s \geq 0$, $\varepsilon = 0$ or $1$. Let $I \subseteq H^{*,*}(\overline{G}, k)$ be a homogeneous ideal stable with respect to the action of the Steenrod operations. Suppose $I$ contains a nonzero element of degree $(2, 0)$. Then one of the following holds:*

(i) *Some element of the form $x_r^n \beta \mathscr{P}^0(v_1) \dots \beta \mathscr{P}^0(v_m)$ (with $n$ and $m$ not both zero) lies in $I$, where $v_1, \dots, v_m$ are nonzero elements of $H^1((\mathbb{Z}/p)^{\times s}, \mathbb{F}_p) \subseteq H^{1,0}(\overline{G}, k)$, or*

(ii) *$I \cap H^{2,0}(\overline{G}, k)$ is one dimensional, spanned by an element of the form $\zeta^2 + \gamma x_r$ with $\gamma \in k$.*

*Proof.* We follow, for the most part, the notation and proof in Proposition 3.6 of [3], with adjustments as appropriate to deal with the extra factor, $(\mathbb{G}_a^-)^\varepsilon$.

Any nonzero element $u$ in $I \cap H^{2,0}(\overline{G}, k)$ has the form

$$u = \alpha \zeta^2 + \sum_{1 \leq i < j \leq r} a_{i,j} \lambda_i \lambda_j + \sum_{1 \leq j \leq r} b_j x_j + \sum_{1 \leq i \leq r, 1 \leq j \leq s} c_{i,j} \lambda_i y_j + \sum_{1 \leq i < j \leq s} d_{i,j} y_i y_j + \sum_{1 \leq j \leq s} e_j z_j$$

for scalars $\alpha, a_{i,j}, b_j, c_{i,j}, d_{i,j}, e_j$ which are not all zero, and the term $\alpha \zeta^2$ only occurs if $\varepsilon = 1$.

First suppose that each such $u$ has $\alpha \neq 0$. In this case $I \cap H^{2,0}(\overline{G}, k)$ is one dimensional and $\varepsilon = 1$. Furthermore, $u$ has to be sent to a multiple of itself by $\mathscr{P}^0$. The Cartan formula implies $\zeta^2$ is killed by $\mathscr{P}^0$, and so is $u$. The condition $\mathscr{P}^0(u) = 0$ forces $u$ to be of the form

$$(4.2) \qquad u = \zeta^2 + \sum_{1 \leq i < r} a_{i,r} \lambda_i \lambda_r + b_r x_r + \sum_{1 \leq j \leq s} c_{r,j} \lambda_r y_j.$$

Assume on the other hand that there exists a $u$ with $\alpha = 0$. Repeated application of $\mathscr{P}^0$ to such a $u$ results in an element of the form $\sum_{1 \leq i < j \leq s} d_{i,j}^{p^\ell} y_i y_j + \sum_{1 \leq j \leq s} e_j^{p^\ell} z_j$. So if at least one of the $d_{i,j}$ or $e_j$ is nonzero, we may apply Theorem 4.2, and this puts us in case (i) with $m > 0$ and $n = 0$. So we may assume

$$u = \sum_{1 \leq i < j \leq r} a_{i,j} \lambda_i \lambda_j + \sum_{1 \leq j \leq r} b_j x_j + \sum_{1 \leq i \leq r, 1 \leq j \leq s} c_{i,j} \lambda_i y_j.$$

Repeatedly applying $\mathscr{P}^0$ and stopping just before we get zero, we can assume $u$ has the form

$$(4.3) \qquad u = \sum_{1 \leq i < r} a_{i,r} \lambda_i \lambda_r + b_r x_r + \sum_{1 \leq j \leq s} c_{r,j} \lambda_r y_j.$$

So we are now in a situation where $u$ has either the form (4.2) or (4.3), and in the first case $I \cap H^{2,0}(G, k)$ is one dimensional. In either case, if some $c_{r,j}$ is nonzero, we apply $\beta \mathscr{P}^0$ to get

$$\beta \mathscr{P}^0(u) = \sum_{1 \leq i < r} a_{i,r}^p \lambda_{i+1} x_r - \sum_{1 \leq j \leq s} c_{r,j}^p x_r y_j \in I.$$

12

Applying $\beta\mathscr{P}^1$, we get

$$\beta\mathscr{P}^1\beta\mathscr{P}^0(u) = -\sum_{1\leq i<r} a_{i,r}^{p^2}x_{i+1}x_r^p - \sum_{1\leq j\leq s} c_{r,j}^{p^2}x_r^p z_j \in I.$$

Now apply $\mathscr{P}^p$ to get

$$\mathscr{P}^p\beta\mathscr{P}^1\beta\mathscr{P}^0(u) = -\sum_{1\leq i<r} a_{i,r}^{p^3}x_{i+2}x_r^{p^2} - \sum_{1\leq j\leq s} c_{r,j}^{p^3}x_r^{p^2} z_j \in I.$$

Successively applying $\mathscr{P}^{p^2}$, $\mathscr{P}^{p^3}$, ..., we eventually conclude that $I$ contains an element of the form $\sum_j c_{r,j}^{p^t} x_r^{p^{t-1}} z_j = x_r^{p^{t-1}}(\sum_j c_{r,j}^{p^t} z_j)$. The set of all such elements in $I$ is stable under the Frobenius map (raising all the coefficients to the $p$th power), and therefore there is a nonzero element with coefficients in $\mathbb{F}_p$. This puts us in case (i) with $m = 1$.

If every $c_{r,j} = 0$ but some $a_{i,r}$ is nonzero, then

$$\beta\mathscr{P}^1\beta\mathscr{P}^0(u) = -\sum_{1\leq i<r} a_{i,r}^{p^2}x_{i+1}x_r^p.$$

Now we apply $\mathscr{P}^p$, then $\mathscr{P}^{p^2}$, and so on, and just before we get zero, we get a multiple of a power of $x_r$. This gives case (i) with $m = 0$ and $n > 1$.

It remains to consider the case when all $c_{r,j}$ and all $a_{i,r}$ are zero. Then, if $u$ has form (4.2) we are in case (ii), and if $u$ has form (4.3) we are in case (i) with $m = 0$ and $n = 1$. $\qquad\square$

To complete the description of the kernel of (4.1), we quote a result from [14] which describes what happens when the kernel of the map $f^*\colon H^{2,*}(\overline{G}, k) \to H^{2,*}(G, k)$ has an element of degree $(2, 1)$. Note that in this case we necessarily have $\epsilon = 1$.

**Theorem 4.4.** *Let $G$ be a finite unipotent supergroup scheme and $N$ a normal sub-supergroup scheme with $G/N \cong \mathbb{G}_a^- \times \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^{\times s}$. If the inflation $H^{1,*}(G/N, k) \to H^{1,*}(G, k)$ is an isomorphism and $H^{2,1}(G/N, k) \to H^{2,1}(G, k)$ is not injective then there exists a nonzero element $\xi \in H^{1,1}(G, k)$ such that $\beta\mathscr{P}^0(u)\xi^{p^{r+s-1}(p-1)} = 0$ for all $u \in H^{1,0}(G, k)$.* $\qquad\square$

## 5. Super Quillen–Venkov

We require an analogue of the Quillen–Venkov lemma ([39]). The proof in [39], and its later variants carry over to the present context; we adapt a purely representation–theoretic approach due to Kroll [32].

*Remark* 5.1. If $H \leq G$ is a maximal sub-supergroup scheme with $G = G^0 \rtimes \pi$ unipotent, then there are three possibilities for $G/H$, namely $\mathbb{G}_{a(1)}$, $\mathbb{Z}/p$, and $\mathbb{G}_a^-$.

- If $G/H \cong \mathbb{G}_{a(1)}$ then there is an element $\lambda \in H^{1,0}(G^0, k)^\pi \subseteq H^{1,0}(G, k)$ corresponding to the homomorphism $G \to \mathbb{G}_{a(1)}$ as in Lemma 3.1, and an associated element $x = -\beta\mathscr{P}^0(\lambda) \in H^{2,0}(G, k)$.
- If $G/H \cong \mathbb{Z}/p$ then there is an element $y \in H^{1,0}(\pi, \mathbb{F}_p) \subseteq H^{1,0}(G, k)$ corresponding to the homomorphism $G \to \mathbb{Z}/p$ as in Lemma 3.1, and an associated element $z = \beta\mathscr{P}^0(y) \in H^{2,0}(G, k)$.
- If $G/H \cong \mathbb{G}_a^-$ then there is an element $\zeta \in H^{1,1}(G, k) \cong \mathrm{Hom}_{\mathsf{sGr}/k}(G, \mathbb{G}_a^-)$ corresponding to the homomorphism $G \to \mathbb{G}_a^-$ as in Lemma 3.1.

13

For $H < G$, we denote by

$$\mathrm{ind}_H^G \colon \mathsf{Mod}\, H \to \mathsf{Mod}\, G$$

the *induction* functor which is the right adjoint to the restriction functor $\mathrm{res}_H^G \colon \mathsf{Mod}\, G \to \mathsf{Mod}\, H$ (note that we follow the group scheme terminology here, as introduced, for example, in [30, I.3]). There is also the coinduction functor

$$\mathrm{coind}_H^G \colon \mathsf{Mod}\, H \to \mathsf{Mod}\, G$$

which is left adjoint to the restriction. In the unipotent case induction and coinduction are canonically isomorphic (see [30, I.3]) which we use implicitly in the proof below.

If $H \leq G$ is a normal subgroup, then kernel of the canonical map $\mathrm{coind}_H^G k \to k$ is the *relative syzygy* $\Omega_{G/H}(k)$. The identity map on the trivial representation $k$ induces a map

(5.1) $$\eta \colon \Omega(k) \to \Omega_{G/H}(k)$$

Similarly, we have a map

$$\eta' \colon \Omega_{G/H}^{-1}(k) \to \Omega^{-1}(k).$$

We employ the same notation $\eta'$ for the shifts of this map.

**Lemma 5.2.** *Let $H \leq G$ be a normal sub-supergroup scheme of a finite unipotent group scheme $G$ with $G/H$ isomorphic to $\mathbb{Z}/p$ or $\mathbb{G}_{a(1)}$. Then $z = \beta \mathscr{P}^0(y)$, respectively $x = -\beta \mathscr{P}^0(\lambda) \in H^{2,0}(G, k)$ (cf. Remark 5.1), is represented by the composite*

$$\Omega(k) \xrightarrow{\ \eta\ } \Omega_{G/H}(k) \xrightarrow{\ \cong\ } \Omega_{G/H}^{-1}(k) \xrightarrow{\ \eta'\ } \Omega^{-1}(k)\ .$$

*Proof.* We prove this in the case where $G/H \cong \mathbb{Z}/p$. The case $G/H \cong \mathbb{G}_{a(1)}$ is proved by replacing $z$ by $x$ everywhere.

The cohomology class $z = \beta \mathscr{P}^0(y) \in \mathrm{Ext}_G^2(k, k)$ is represented by the extension

(5.2) $$0 \longrightarrow k \longrightarrow \mathrm{ind}_H^G k \longrightarrow \mathrm{ind}_H^G k \longrightarrow k \longrightarrow 0$$

This follows from the fact that $\mathrm{ind}_H^G k \cong kG/H = k\mathbb{Z}/p$, and this sequence is the inflation of the extension

$$0 \longrightarrow k \longrightarrow k\mathbb{Z}/p \longrightarrow k\mathbb{Z}/p \longrightarrow k \longrightarrow 0$$

of $\mathbb{Z}/p$ representing corresponding cohomology class (see, for example, [7, I.3.4.2]).

$$0 \longrightarrow \Omega^2(k) \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k \longrightarrow 0$$

(diagram)

$$\Omega(k)$$
$$\downarrow$$
$$\Omega_{G/H}(k)$$

$$0 \longrightarrow k \longrightarrow \mathrm{ind}_H^G k \longrightarrow \mathrm{ind}_H^G k \longrightarrow k \longrightarrow 0$$

$$\Omega_{G/H}^{-1}(k)$$
$$\downarrow$$
$$\Omega^{-1}(k)$$

$$0 \longrightarrow k \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \Omega^{-2}(k) \longrightarrow 0$$

with vertical maps $z$, $1$, $\cong$, $z$ and $1$.

By (5.2) the sequence in the middle row represents $z$. So comparing with a projective resolution as in the top row, the comparison map $\Omega^2(k) \to k$ represents $z$. Dually, comparing with an injective resolution as in the bottom row, the comparison map $k \to \Omega^{-2}(k)$ also represents $z$. Therefore the vertical composite map in the middle of the diagram also represents $z$. $\qquad\square$

Given $\xi \in H^{s,t}(G,M)$, for each $n \geq 0$ we write $\xi^n$ for the class $\xi^{\otimes n} \in H^{ns,nt}(G, M^{\otimes n})$.

**Proposition 5.3.** *Let $H$ be a maximal sub-supergroup scheme of a finite supergroup scheme $G$, and let $M$ be a $G$-module. Suppose that $\xi \in H^{*,*}(G,M)$ restricts to zero on $H$. Then*

(i) *if $G/H \cong \mathbb{Z}/p$ then $\xi^2$ is divisible by the element $z = \beta \mathscr{P}^0(y) \in H^{2,0}(G,k)$,*

(ii) *if $G/H \cong \mathbb{G}_{a(1)}$ then $\xi^2$ is divisible by the element $x = -\beta \mathscr{P}^0(\lambda) \in H^{2,0}(G,k)$,*

(iii) *if $G/H \cong \mathbb{G}_a^-$ then $\xi$ is divisible by the element $\zeta \in H^{1,1}(G,k)$.*

*Proof.* We shall start by proving (ii). Let $\xi \in H^{n,*}(G,M)$, and choose a map $\Omega^n(k) \to M$ representing $\xi$; by abuse of notation we call this map $\xi$. We also use $\xi$ to denote any shift of this map, as a map from $\Omega^{n+i}(k)$ to $\Omega^i(M)$ for $i \in \mathbb{Z}$.

The exact sequence $k \xrightarrow{\epsilon'} \mathrm{ind}_H^G k \to \Omega_{G/H}^{-1}(k)$ induces a triangle in $\mathsf{stmod}(G)$

$$M \otimes \Omega(\Omega_{G/H}^{-1}(k)) \xrightarrow{1 \otimes \eta'} M \xrightarrow{1 \otimes \epsilon'} M \otimes \mathrm{ind}_H^G k \to M \otimes \Omega_{G/H}^{-1}(k).$$

The assumption that $\xi$ restricts to zero on $H$ means that the restriction of $\xi \colon \Omega^n(k) \to M$ to $H$ factors through a projective. Hence, so does the adjoint map

$$\Omega^n(k) \to \mathrm{ind}_H^G M = M \otimes \mathrm{ind}_H^G k.$$

This adjoint factors as the composite of $\xi$ with $1 \otimes \epsilon'$. The fact that this composition factors through a projective implies that there exists a lifting $\rho' \colon \Omega^n(k) \to \Omega(\Omega_{G/H}^{-1}(k))$ making the following diagram commute:

$$
\begin{array}{c}
\Omega^n(k) \\
\swarrow_{\rho'} \quad \downarrow_{\xi} \quad \searrow \\
M \otimes \Omega(\Omega_{G/H}^{-1}(k)) \xrightarrow{1 \otimes \eta'} M \xrightarrow{1 \otimes \epsilon'} M \otimes \mathrm{ind}_H^G k \longrightarrow M \otimes \Omega_{G/H}^{-1}(k).
\end{array}
$$

15

Shifting by $\Omega^{-1}$, we get a commutative diagram

$$(5.3)$$

$$
\begin{array}{ccc}
 & & \Omega^{n-1}(k) \\
 & \nearrow^{\rho'} & \downarrow{\xi} \\
M \otimes \Omega^{-1}_{G/H}(k) & \xrightarrow{1\otimes\eta'} & M \otimes \Omega^{-1}(k).
\end{array}
$$

Similarly, we can factor $\xi\colon \Omega(k) \to M \otimes \Omega^{-n+1}(k)$ to obtain a commutative diagram

$$
\begin{array}{ccc}
\Omega(k) & \xrightarrow{\eta} & \Omega_{G/H}(k) \\
\downarrow{\xi} & \swarrow{\rho} & \\
M \otimes \Omega^{-n+1}(k) & &
\end{array}
$$

Tensoring with $M$, we get a commutative diagram

$$(5.4)$$

$$
\begin{array}{ccc}
M \otimes \Omega(k) & \xrightarrow{1\otimes\eta} & M \otimes \Omega_{G/H}(k) \\
\downarrow{1\otimes\xi} & \swarrow{1\otimes\rho} & \\
M \otimes M \otimes \Omega^{-n+1}(k) & &
\end{array}
$$

Putting (5.3) and (5.4) together, we get the following diagram, where the composite of the maps in the middle row is $1 \otimes x$ by Lemma (5.2):

$$
\begin{array}{ccccccc}
 & & & & & & \Omega^{n-1}(k) \\
 & & & & & \nearrow^{\rho'} & \downarrow{\xi} \\
M \otimes \Omega(k) & \xrightarrow{1\otimes\eta} & M \otimes \Omega_{G/H}(k) & \xrightarrow{\cong} & M \otimes \Omega^{-1}_{G/H}(k) & \xrightarrow{1\otimes\eta'} & M \otimes \Omega^{-1}(k) \\
\downarrow{1\otimes\xi} & \swarrow{1\otimes\rho} & & & & & \\
M \otimes M \otimes \Omega^{-n+1}(k) & & & & & &
\end{array}
$$

with $1\otimes x$ the curved top arrow.

Completing the diagram, we see that $\xi^2 = (1 \otimes \xi) \circ \xi$ factors through $x$ either on the left or on the right.

$$
\begin{array}{ccccccc}
\Omega^{n+1}(k) & \xrightarrow{\quad\quad x \quad\quad} & & & & & \Omega^{n-1}(k) \\
\downarrow{\xi} & & & & & \nearrow^{\rho'} & \downarrow{\xi} \\
M \otimes \Omega(k) & \xrightarrow{1\otimes\eta} & M \otimes \Omega_{G/H}(k) & \xrightarrow{\cong} & M \otimes \Omega^{-1}_{G/H}(k) & \xrightarrow{1\otimes\eta'} & M \otimes \Omega^{-1}(k) \\
\downarrow{1\otimes\xi} & \swarrow{1\otimes\rho} & & & & & \downarrow{1\otimes\xi} \\
M \otimes M \otimes \Omega^{-n+1}(k) & & & \xrightarrow{1\otimes1\otimes x} & & & M \otimes M \otimes \Omega^{-n-1}(k).
\end{array}
$$

The same argument works for part (i). Part (iii) is similar but easier. Namely, we have a short exact sequence of $kG$-modules

$$0 \to k \to \mathrm{ind}_H^G k \xrightarrow{\varepsilon} k \to 0.$$

We have failed to distinguish whether $k$ is in even or odd degree, but the two ends are in opposite degrees. The connecting map for this in $\mathsf{stmod}(kG)$ is $\zeta$, so we have a triangle

$$k \to \mathrm{ind}_H^G k \xrightarrow{\varepsilon} k \xrightarrow{\zeta} \Omega^{-1}(k).$$

If $\xi\colon k \to \Omega^{-n}(M)$ restricts to the zero class on $H$ then the composite with $\mathrm{ind}_H^G k \xrightarrow{\varepsilon} k$ is zero, and so $\xi$ factors through $\zeta$. $\qquad\square$

## 6. Nilpotence and projectivity

**Definition 6.1.** Let $G$ be a finite supergroup scheme and $M$ be a $G$-module. We say that a class $\xi \in H^{j,*}(G, M)$ is *nilpotent* if there exists $n \geq 1$ such that $\xi^n \in H^{jn,*}(G, M^{\otimes n})$ is zero.

In the remainder of the paper we employ the following terminology. Let $G$ be a finite supergroup scheme, and let $\mathcal{H}$ be a *family of subgroups after field extension*, namely a family of pairs $(H, K)$ where $K$ is an extension field of $k$ and $H$ is a sub supergroup scheme of $G_K$. Note that the embeddings of $H$ in $G_K$ *need not* be defined over the ground field $k$.

We say that nilpotence of cohomology elements is *detected* on the family $\mathcal{H}$ if for any $G$-module $M$ and cohomology class $\xi \in H^{*,*}(G, M)$, we have that $\xi$ is nilpotent if and only if $\mathrm{res}_H^{G_K}(\xi_K) \in H^{*,*}(H, M_K)$ is nilpotent for every $(H, K) \in \mathcal{H}$.

Similarly, we say that projectivity of modules is *detected* on the family $\mathcal{H}$ if for any $G$-module $M$, we have that $M$ is projective if and only if $\mathrm{res}_H^{G_K}(M_K)$ is projective as an $H$-module for every $(H, K) \in \mathcal{H}$.

In particular, we say that nilpotence and projectivity are detected on proper subgroups of $G$ after field extensions if the family $\mathcal{H}$ can be taken to be the family of all pairs $(H, K)$ where $K$ runs over all field extensions of $k$ and $H$ runs over all proper subgroups of $G_K$. In practice, it always suffices to take $K$ to be an algebraically closed field of large enough finite transcendence degree over $k$.

**Lemma 6.2.** *Let $G$ be a finite supergroup scheme, $M$ a $G$-module, and fix an element $\xi \in H^{j,*}(G, M)$ with $j > 0$. With $\xi\colon k \to \Omega^{-j}(M)$ denoting also corresponding map on modules, let $X$ be the colimit*

$$X = \mathrm{colim}\{\, k \xrightarrow{\ \xi\ } \Omega^{-j}(M) \xrightarrow{1 \otimes \xi} \Omega^{-2j}(M^{\otimes 2}) \xrightarrow{1 \otimes 1 \otimes \xi} \Omega^{-3j}(M^{\otimes 3}) \longrightarrow \cdots \}$$

*Then $\xi$ is nilpotent if and only if $X$ is projective.*

*Proof.* If $\xi^n = 0$, then the composite of any $n$ consecutive maps in the system defining $X$ factors through a projective, and so $X$ is projective. Conversely, if $X$ is projective, then the map $k \to X$ factors through a projective. Since $k$ is finite dimensional, it factors through a finite dimensional projective, and hence a finite composite of maps in the defining system factors through a projective. This implies that the corresponding power of $\xi$ is zero. $\qquad\square$

Lemma 6.2 immediately implies the following result.

**Theorem 6.3.** *Let $G$ be a finite supergroup scheme. If a family $\mathcal{H}$ of proper sub supergroup schemes after field extensions detects projectivity of $G$-modules, then it also detects nilpotence of cohomology elements.*

*Proof.* Let $M$ be a $G$-module and $\xi \in H^{j,*}(G, M)$ an element with $j > 0$. Represent it by a map $\xi \colon k \to \Omega^{-j}(M)$, and consider the colimit $X = \operatorname{colim} \Omega^{-jn}(M^{\otimes n})$ as in Lemma 6.2.

Our assumption is that $\xi_K\!\downarrow_H$ is nilpotent for each $(H, K) \in \mathcal{H}$. That is, for some $n$ depending on $(H, K)$, the element $(\xi_K\!\downarrow_H)^{\otimes n} \in H^{jn,*}(H, M_K^{\otimes n})$ is zero. Equivalently, the map $K \to \cdots \to \Omega^{-jn}(M^{\otimes n})$ factors through a projective upon restriction to $H$. Hence, $X_K\!\downarrow_H$ is projective. Since we assumed that projectivity is detected on the family $\mathcal{H}$, we conclude that $X$ is projective $G$-module. The statement now follows by Lemma 6.2. $\qquad\square$

We omit the proof of the following lemma since the proof is similar to [11, Lemma 3.5] if one replaces $\pi$-support with the cohomological support. See also [15].

**Lemma 6.4.** *Let $G$ be a finite supergroup scheme, and $M$ be a $G$-module. The following are equivalent:*
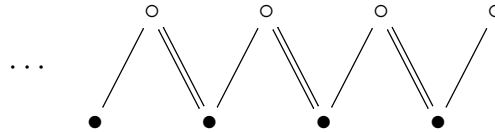
(a) *$M$ is projective,*
(b) *any class $\xi \in \operatorname{Ext}_G^{>0,*}(M, M)$ is nilpotent.* $\qquad\square$

Here is a partial converse to Theorem 6.3.

**Proposition 6.5.** *Let $G$ be a finite supergroup scheme. Suppose that nilpotence in cohomology of $G$-modules is detected on a family $\mathcal{H}$ of proper subgroups of $G$ without field extension (i.e., each pair $(H, K) \in \mathcal{H}$ has $K = k$). Then projectivity of modules is also detected on $\mathcal{H}$.*

*Proof.* Let $N$ be a $G$-module such that $N\!\downarrow_H$ is projective for all $H \in \mathcal{H}$. Then $\Lambda = \operatorname{End}_k(N)$ is projective upon restriction to each $H \in \mathcal{H}$ so that for any cohomology class $\xi \in H^{*,*}(G, \Lambda)$, we have $\xi\!\downarrow_H = 0$. Since nilpotency is detected on $\mathcal{H}$ we deduce that all elements $\xi \in H^{>0,*}(G, \Lambda) \cong \operatorname{Ext}_G^{>0,*}(N, N)$ are nilpotent. Now apply Lemma 6.4. $\qquad\square$

*Remark* 6.6. The full converse to Theorem 6.3 is trickier. The argument above fails if we have to extend scalars. A deeper reason might be that it is not true that nilpotency of all elements in $H^*(G, M)$ implies that $M$ is projective. We refer the reader to a cautionary example described in Proposition 5.1 of [8]: take $G$ to be the Klein group $\mathbb{Z}/2 \times \mathbb{Z}/2$, $p = 2$, and $M$ be an infinite dimensional module represented by the following diagram:



As computed in [8, Proposition 5.1], all cohomology classes in $H^*(G, M)$ are nilpotent (of unbounded degree) whereas the module $M$ is not projective.

## 7. Inductive detection theorem

We finish the first part of the paper with the inductive detection theorem. The point Theorem 7.2 is to cover the "easy" cases of the detection, leaving the task of showing that the finite unipotent supergroup schemes not covered by Hypotheses 7.1 are precisely the elementary supergroup schemes as defined in (1.1), see Theorem 11.1. It is in the preparation work for that theorem the degree 2 cohomology element of Theorem 4.3 becomes relevant.

We separate out the hypotheses since these will appear again in Section 11.

**Hypothesis 7.1.** The finite supergroup scheme $G$ is unipotent and satisfies at least one of the following:

  (a) There is a surjective map $G \to \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$.
  (b) There is a surjective map $G \to \mathbb{G}_a^- \times \mathbb{G}_a^-$.
  (c) There are nonzero elements

$$\lambda_1, \ldots \lambda_n \in H^{1,0}(G^0, k)^\pi \subseteq H^{1,0}(G, k)$$
$$y_1, \ldots, y_s \in H^{1,0}(\pi, \mathbb{F}_p) \subseteq H^{1,0}(G, k)$$
$$\zeta_1, \ldots, \zeta_m \in H^{1,1}(G, k) \cong \mathrm{Hom}(G, \mathbb{G}_a^-)$$

such that $\prod \beta \mathscr{P}^0(\lambda_i) \prod \beta \mathscr{P}^0(y_j) \prod \zeta_\ell = 0$.

**Theorem 7.2.** *If Hypothesis 7.1 hold for $G$, then*

  (i) *nilpotence of elements of $H^{*,*}(G, M)$ and*
  (ii) *projectivity of $kG$-modules*

*are detected on proper sub-supergroup schemes after field extension.*

*Proof.* The argument that if $G$ satisfies either condition (a) or (b), then projectivity of modules is detected on proper sub-supergroup schemes goes exactly as in the case 3(b) of the proof of [3, Theorem 8.1] so we will not reproduce it here. The main ingredient of the proof is the Kronecker quiver lemma, see [9, Lemma 4.1]. Once we know detection of projectivity, the detection of nilpotents is implied by Theorem 6.3.

We now show that (c) implies detection of nilpotents in $H^{*,*}(G, M)$ on sub-supergroup schemes, without any field extensions. Let $\xi \in H^{n,*}(G, M)$ be a cohomology class which restricts nilpotently to all proper subgroups of $G$, and let $\prod \beta \mathscr{P}^0(\lambda_i) \prod \beta \mathscr{P}^0(y_j) \prod \zeta_\ell = 0$. Each of the elements $\lambda_i$, $y_j$, $\zeta_\ell$ corresponds to a map from $G$ to $\mathbb{Z}/p$, $\mathbb{G}_{a(1)}$ or $\mathbb{G}_a^-$, with $\xi$ restricting nilpotently to the kernel of the corresponding map. Proposition 5.3 implies that $\xi^{2i+2j+\ell}$ is then divisible by $\prod \beta \mathscr{P}^0(\lambda_i) \prod \beta \mathscr{P}^0(y_j) \prod \zeta_\ell$, and is therefore zero.

Finally, since the case (c) does not involve field extensions, Proposition 6.5 implies that we also have detection of projectivity in this case. $\square$

# Part 2. The detection theorem

## 8. WITT ELEMENTARY SUPERGROUP SCHEMES

In this section we introduce a family of Witt elementary supergroup schemes that play an essential role in our main detection theorem.

**Notation 8.1.** We shall make an extensive use of diagrams to depict many of the unipotent connected supergroup schemes to be introduced in this section. In these diagrams, $\circ$ denotes a composition factor isomorphic to $\mathbb{G}_{a(1)}$ and $\bullet$ denotes a composition factor isomorphic to $\mathbb{G}_a^-$. A single bond represents an extension of $\mathbb{G}_{a(1)}$ by $\mathbb{G}_{a(1)}$ to make $\mathbb{G}_{a(2)}$ and the double bond represents an extension of $\mathbb{G}_{a(1)}$ by $\mathbb{G}_{a(1)}$ to make $W_{2,1}$. The dashed link denotes an

extension of $\mathbb{G}_a^-$ by $\mathbb{G}_{a(1)}$ to make the supergroup scheme $W_{1,1}^-$ discussed in Example 2.12.



$$\mathbb{G}_{a(2)} \qquad W_{2,1} \qquad W_{1,1}^-$$
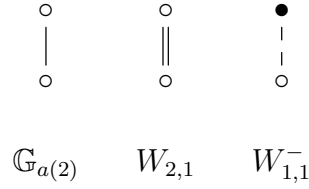
**Example 8.2.** Let $\mathfrak{g}$ be the $p$-restricted Lie superalgebra described in Example 5.3.3 of Drupieski and Kujawa [23]. This is generated by an odd degree element $\sigma$ and an even degree element $s$ satisfying $[\sigma, \sigma] = 2s^{[p]}$. This is unipotent if and only if some $s^{[p^m]}$ is zero. If $n$ is minimal with this property then $\mathfrak{g}$ has a basis consisting of $\sigma, s, s^{[p]}, \ldots, s^{[p^{m-1}]}$. The restricted enveloping algebra of $\mathfrak{g}$ is the group algebra of the finite supergroup scheme denoted $E_{m,1}^-$ with

$$kE_{m,1}^- = k[s, \sigma]/(s^{p^m}, \sigma^2 - s^p)$$

where $s$ and $\sigma$ are primitive. Note that $(E_{m,1}^-)_{\mathsf{ev}} \cong W_{m,1}$, the first Frobenius kernel of length $m$ Witt vectors as introduced in the Appendix, so we have a short exact sequence
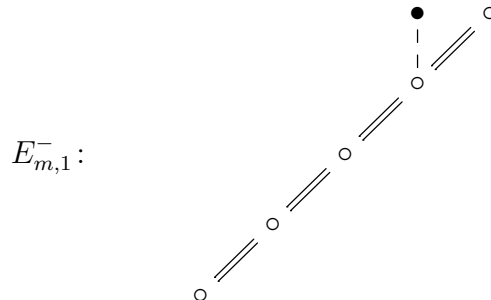
(8.1) $$1 \to W_{m,1} \to E_{m,1}^- \to \mathbb{G}_a^- \to 1.$$

For $m \geq 2$, there are also short exact sequences
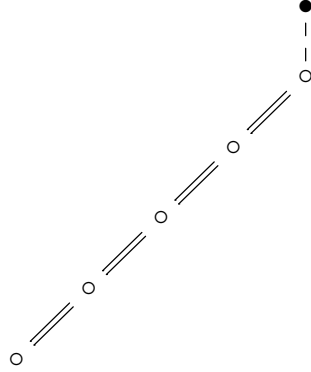
$$1 \to W_{m-1,1}^- \to E_{m,1}^- \to \mathbb{G}_{a(1)} \to 1,$$

where $kW_{m-1,1}^- = k[\sigma]/\sigma^{2p^{m-1}}$ (see Example 2.12), and

$$1 \to W_{m-1,1} \to E_{m,1}^- \to \mathbb{G}_{a(1)} \times \mathbb{G}_a^- \to 1$$

where the group algebra of $W_{m-1,1}$ is generated by $s^p = \sigma^2$. Using Notation 8.1, $E_{m,1}^-$ is represented with the following diagram.

$$E_{m,1}^-:$$

As another example, we draw a diagram for $W_{m,1}^-$ of Example 2.12.

Lemma 8.3. *If $G$ is a finite supergroup scheme which sits in a short exact sequence*

$$1 \to \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)} \to G \to \mathbb{G}_a^- \to 1$$

*then there is a non-trivial homomorphism $G \to \mathbb{G}_{a(1)}$.*

*Proof.* By Corollary 2.8, the height of $G$ is one so it is of the form $\mathcal{U}^{[p]}(\mathfrak{g})$ with $\mathfrak{g} = \mathsf{Lie}(G)$. Then $\mathfrak{g}$ has a two dimensional even part with trivial commutator and $p$-restriction map, and a one dimensional odd part. There is therefore a non-trivial homomorphism from $\mathfrak{g}$ to the one dimensional trivial Lie algebra $\mathsf{Lie}(\mathbb{G}_{a(1)})$, and this induces a non-trivial homomorphism from $G$ to $\mathbb{G}_{a(1)}$. $\square$

Next we classify all extensions of $\mathbb{G}_a^-$ by $W_{m,1}$ complementing examples (2.3) and (8.1).
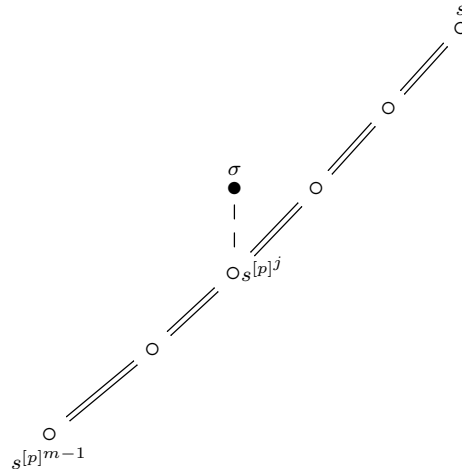
Lemma 8.4. *Let $G$ be a finite supergroup scheme fitting in an extension*

$$1 \to W_{m,1} \to G \to \mathbb{G}_a^- \to 1.$$

*Then*

$$kG \cong k[s, \sigma]/(s^{p^m}, \sigma^2 - s^{p^j})$$

*for some $0 \leq j \leq m-1$, where $\sigma$ is odd, $s$ is even, and both are primitive. Hence, $G$ can be represented by the following picture:*

*Proof.* By assumption, $G_{\mathsf{ev}} = W_{m,1}$. Hence, $G$ has height 1 by Theorem 2.6. By Lemma 2.14, there is a Lie superalgebra $\mathfrak{g}$ such that $\mathcal{U}^{[p]}(\mathfrak{g}) \cong kG$. Let $\sigma$ be a lifting to $\mathfrak{g}$ of the generator of $k\mathbb{G}_a^-$, and let $s$ be an algebraic generator of $kW_{m,1}$, that is, $s, s^{[p]}, \ldots, s^{[p]^{m-1}}$ be a basis of the Lie algebra corresponding to $W_{m,1}$. Then we have

$$\tfrac{1}{2}[\sigma, \sigma] = \sum_0^{m-1} a_i s^{[p]^i}.$$

Let $j$ be the minimal index such that $a_j \neq 0$ and set $s' = \sum_j^{m-1} a_i s^{[p]^{i-j}}$. The generators $\sigma, s'$ give the asserted presentation of $\mathcal{U}^{[p]}(\mathfrak{g}) \cong kG$. $\square$

**Construction 8.5** ($E_{m,n}^-$)**.** There is a homomorphism $E_{m,1}^- \to \mathbb{G}_{a(1)}$ given by factoring out the ideal of $kE_{m,1}^-$ generated by $\sigma$. There is also a surjective map $\mathbb{G}_{a(n)} \to \mathbb{G}_{a(1)}$ given by the $(n-1)$st power of the Frobenius map. We define $E_{m,n}^-$ to be the kernel of the map from the product to $\mathbb{G}_{a(1)}$, so that there is a short exact sequence

$$1 \to E_{m,n}^- \to E_{m,1}^- \times \mathbb{G}_{a(n)} \to \mathbb{G}_{a(1)} \to 1.$$

Its group ring is given by

$$kE_{m,n}^- = k[s_1, \ldots, s_{n-1}, s_n, \sigma]/(s_1^p, \ldots, s_{n-1}^p, s_n^{p^m}, \sigma^2 - s_n^p)$$

where $s_1, \ldots, s_n$ are in even degree and $\sigma$ is in odd degree. The comultiplication is given by

$$\Delta(s_i) = S_{i-1}(s_1 \otimes 1, \ldots, s_i \otimes 1, \ 1 \otimes s_1, \ldots, 1 \otimes s_i) \qquad (i \geq 1)$$
$$\Delta(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma$$

where the $S_i$ are as defined in Appendix A, and come from the comultiplication in $\mathrm{Dist}(\mathbb{G}_a)$.
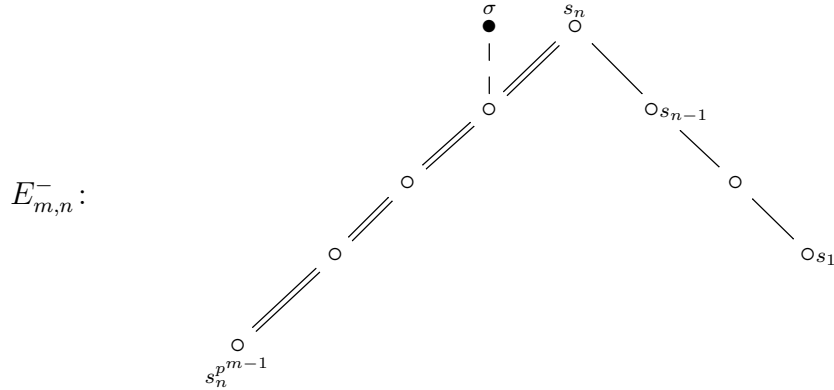


We define

$$E_{m,n} := (E_{m,n})_{\mathsf{ev}}^-$$

and observe that there is an isomorphism

$$E_{m,n} \cong W_{m,n}/W_{m-1,n-1}.$$

**Definition 8.6.** A finite supergroup scheme is *Witt elementary* if it is isomorphic to a quotient of $E_{m,n}^-$ by an even subgroup scheme.

22

*Remark* 8.7. For $m = 1$, $E^-_{m,n}$ splits as a direct product:

$$E^-_{1,n} \cong \mathbb{G}_{a(n)} \times \mathbb{G}^-_a$$

**Lemma 8.8.** *Let $G$ be a finite supergroup scheme with the connected component $G^0$ and the group of connected components $\pi = \pi(G)$ which is a p-group. If $G^0$ is an extension*

$$1 \to \mathbb{G}_{a(1)} \to G^0 \to \mathbb{G}^-_a \to 1$$

*then $G = G^0 \times \pi$.*

*Proof.* Since $G^0$ has height, it corresponds to a 2-dimensional Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ by Lemma 2.14. Each part is 1-dimensional and must be stabilized by $\pi$. Since $\pi$ is a $p$-group, it centralizes both $\mathbb{G}^-_a$ and $\mathbb{G}_{a(1)}$; hence, centralizes $G$. $\square$

**Lemma 8.9.** *If $G^0 = \mathbb{G}_{a(r)} \times \mathbb{G}^-_a$, and $\pi(G)$ is a p-group, then the subgroup $\mathbb{G}^-_a$ is centralized by $\pi(G)$.*

*Proof.* We must have that $G_{(1)} = \mathbb{G}_{a(1)} \times \mathbb{G}^-_a$ is centralized by $\pi$. Now apply Lemma 8.8. $\square$

**Construction 8.10** ($E^-_{m,n,\mu}$). The group algebra of $E^-_{m+1,n+1}$ is described in Construction 8.5 except that we shift the indexing on the even generators $s_i$ down by 1. With that shift, it has the form

$$kE^-_{m+1,n+1} = k[s_0, s_1, \ldots, s_{n-1}, s_n, \sigma]/(s_0^p, s_1^p, \ldots, s_{n-1}^p, s_n^{p^{m+1}}, \sigma^2 - s_n^p).$$

Let $k\mathbb{G}_{a(1)} = k[s]/s^p$ with $s$ primitive in even degree. For $\mu \in k$, define the supergroup scheme $E^-_{m,n,\mu}$ to be the quotient of $E^-_{m+1,n+1}$ given by the embedding $\mathbb{G}_{a(1)} \to E^-_{m,n,\mu}$ which sends $s$ to $s_0 - m s_n^{p^m}$. Thus, there is a short exact sequence

$$1 \to \mathbb{G}_{a(1)} \to E^-_{m+1,n+1} \to E^-_{m,n,\mu} \to 1.$$

In the language of Dieudonné module introduced in the Appendix, $E^-_{m,n,\mu}$ is quotient of $E^-_{m+1,n+1}$ by the subgroup scheme of $(E^-_{m+1,n+1})_{\mathsf{ev}} \cong \psi(D_k/(V^{m+1}, F^{n+1}, p))$ given by applying $\psi$ to the submodule of $D_k/(V^{m+1}, F^{n+1}, p)$ spanned by $F^n - \mu V^m$. Explicitly, the group ring $kE^-_{m,n,\mu}$ is given by

$$kE^-_{m,n,\mu} = k[s_1, \ldots, s_{n-1}, s_n, \sigma]/(s_1^p, \ldots, s_{n-1}^p, s_n^{p^{m+1}}, \sigma^2 - s_n^p)$$

where $s_1, \ldots, s_n$ are in even degree and $\sigma$ is in odd degree. The comultiplication is given by

$$\Delta(s_i) = S_i(\mu s_n^{p^m} \otimes 1, s_1 \otimes 1, \ldots, s_i \otimes 1, \ 1 \otimes \mu s_n^{p^m}, 1 \otimes s_1, \ldots, 1 \otimes s_i)$$
$$\Delta(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma.$$



23
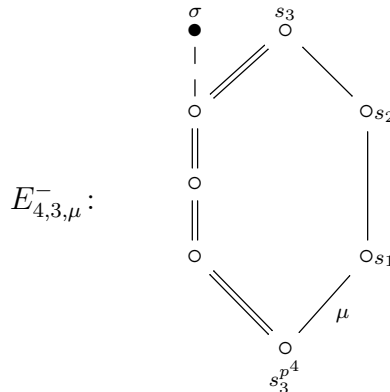
We define
$$E_{m,n,\mu} := (E^-_{m,n,\mu})_{\mathsf{ev}}.$$

**Lemma 8.11.** *Let $G$ be a finite unipotent supergroup scheme.*

(1) *If for some $n \geq 2$, there is an extension*
$$1 \to E_{m,n} \to G \to \mathbb{G}^-_a \to 1,$$
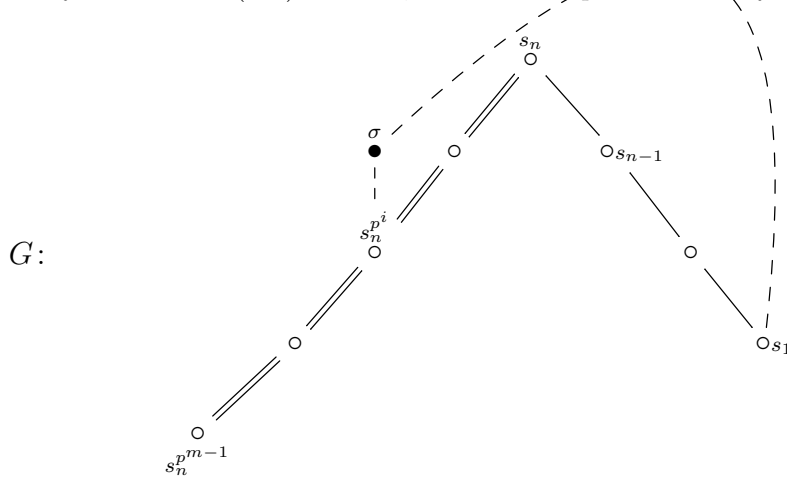*then $kG \cong k[s_1, \ldots, s_{n-1}, s_n, \sigma]/(s_1^p, \ldots, s_{n-1}^p, s_n^{p^m}, \sigma^2 - s_n^{p^i} - \alpha s_1)$ for some $1 \leq i \leq m-1$ and $\alpha \in k$, where $s_1, \ldots, s_n$ are in even degree, $\sigma$ is in odd degree, and comultiplication is given by the formulas in (8.5). Hence, $G$ can be represented as follows:*



(2) *If $G$ fits in the extension*
$$1 \to E_{m,n,\mu} \to G \to \mathbb{G}^-_a \to 1,$$
*then $kG = k[s_1, \ldots, s_{n-1}, s_n, \sigma]/(s_1^p, \ldots, s_{n-1}^p, s_n^{p^{m+1}}, \sigma^2 - s_n^{p^i})$ for some $1 \leq i \leq m - 1$ and degrees and comultiplication as in (8.10).*

*Proof.* We handle only the first case; the second one is similar. We have $(E_{m,n})_{(1)} = W_{m-1,1} \times \mathbb{G}_{a(1)}$, and hence $G_{(1)}$ fits into a short exact sequence:
$$1 \to W_{m-1,1} \times \mathbb{G}_{a(1)} \to G_{(1)} \to \mathbb{G}^-_a \to 1.$$

Let $\mathfrak{g} = \mathsf{Lie}(G)$, so that by Lemma 2.14 we have $kG_{(1)} \cong \mathcal{U}^{[p]}(\mathfrak{g})$. Let $\sigma$ be a lift to $\mathfrak{g}$ of a generator for $\mathsf{Lie}(\mathbb{G}^-_a)$. Then $\sigma$ has odd degree, and $\frac{1}{2}[\sigma, \sigma]$ is some element of $\mathsf{Lie}(W_{m-1,1} \times \mathbb{G}_{a(1)})$, which is the linear span of the elements
$$s_n^{[p]}, s_n^{[p]^2}, \ldots, s_n^{[p]^{m-1}}, s_1 \in \mathsf{Lie}(E_{m,r}).$$
Arguing exactly as in the proof of Lemma 8.4, we can change the generator $s_n$ so that $\frac{1}{2}[\sigma, \sigma] = s_n^{[p]^i} + \alpha s_1$ without changing the comultiplication on $kE_{m,n}$. $\qquad \square$

*Remark* 8.12. The finite supergroup schemes $E^-_{m,n}$ and $E^-_{m,n,\mu}$ also appear in the work of Drupieski and Kujawa [24], where they are denoted $\mathbb{M}_{n;m}$ and $\mathbb{M}_{n+1;m,-\mu}$ respectively.

We also record the structure of the coordinate rings $k[E^-_{m,n}]$ and $k[E^-_{m,n,\mu}]$. For $k[E^-_{m,n}]$ we have generators $w, x_1, \ldots, x_{p^m-1}, y$ with $y$ odd and the remaining generators even. We

have relations $w^{p^{n-1}} = x_1$, $x_i x_j = \binom{i+j}{j} x_{i+j}$, $y^2 = 0$; which implies that as an algebra it is a truncated polynomial ring generated by $w, x_p, x_{p^2}, \dots, x_{p^{m-1}}, y$ with relations $w^{p^n}, x_p^p, x_{p^2}^p, \dots, x_{p^{m-1}}^p, y^2$.

For the coalgebra structure, the elements $w$ and $y$ are primitive, while

$$\Delta(x_\ell) = \sum_{i+j=\ell} x_i \otimes x_j + \sum_{i+j+p=\ell} x_i y \otimes x_j y.$$

The antipode negates $w$ and $y$, and sends $x_i$ to $(-1)^i x_i$.

The coordinate ring $k[E_{m,n,\mu}^-]$ is the subalgebra of $k[E_{m+1,n+1}^-]$ generated by the elements $w - \mu x_{p^m}, x_p, x_{p^2}, \dots, x_{p^{m-1}}, y$ with the restriction of the comultiplication and antipode.

**Theorem 8.13.** *Every Witt elementary supergroup scheme is isomorphic to one of the following:*

    (i) $\mathbb{G}_a^-$,
    (ii) $E_{m,n}^-$ *with* $m, n \geq 1$,
    (iii) $E_{m,n,\mu}^-$ *with* $m, n \geq 1$ *and* $0 \neq \mu \in k$.

*The only isomorphisms between these are given by $E_{m,n,\mu}^- \cong E_{m,n,\mu'}^-$ if and only if $\mu/\mu' = a^{p^{m+n}-1}$ for some $a \in k$.*

*Note that $E_{1,n}^-$ is isomorphic to $\mathbb{G}_{a(n)} \times \mathbb{G}_a^-$ for $n \geq 1$.*

*Proof.* The quotient of $E_{m,n}^-$ by its entire even part is covered in part (i). The quotient by a proper subgroup of $(E_{m,n}^-)_{\mathrm{ev}}$ uses Theorem A.3, and gives parts (ii) and (iii). $\qquad\square$

We recall Definition 1.1 from the Introduction: a finite supergroup scheme is *elementary* if it is isomorphic to a quotient of $E_{m,n}^- \times (\mathbb{Z}/p)^{\times s}$.

*Remark* 8.14. An elementary finite supergroup scheme is isomorphic to one of the following:

    (i) $\mathbb{G}_{a(n)} \times (\mathbb{Z}/p)^{\times s}$ with $n, s \geq 0$,
    (ii) $\mathbb{G}_{a(n)} \times \mathbb{G}_a^- \times (\mathbb{Z}/p)^{\times s}$ with $n, s \geq 0$,
    (iii) $E_{m,n}^- \times (\mathbb{Z}/p)^{\times s}$ with $m \geq 1$, $n \geq 2$, $s \geq 0$, or
    (iv) $E_{m,n,\mu}^- \times (\mathbb{Z}/p)^{\times s}$ with $m, n \geq 1$, $0 \neq \mu \in k$ and $s \geq 0$.

**Definition 8.15.** The *rank* of an elementary finite supergroup scheme is defined to be $n + s$ in case (i), and $n + s + 1$ in cases (ii)–(iv) of the above remark.

## 9. Cohomological calculations

This section is dedicated to computing the cohomology rings of the supergroup schemes introduced in Section 8, and other preparatory results for use in the sequel.

**Proposition 9.1.** *If $G$ is a semidirect product $(\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}) \rtimes (\mathbb{Z}/p)^{\times s}$ with non-trivial action then there is an element $0 \neq y \in H^1((\mathbb{Z}/p)^{\times s}, k) \subseteq H^{1,0}(G, k)$ whose product with $0 \neq \lambda \in H^1(\mathbb{G}_{a(1)}, k) \subseteq H^{1,0}(G, k)$ is zero in $H^{2,0}(G, k)$.*

*Proof.* The non-triviality of the product of a pair of elements in $H^1(G, k) \cong \mathrm{Ext}^1_{kG}(k, k)$ is the obstruction to producing a three dimensional module using these two extensions. So the

proposition follows from the fact that $G$ has a representation of the form

$$\begin{pmatrix} 1 & (\mathbb{Z}/p)^{\times s} & \mathbb{G}_{a(1)} \\ 0 & 1 & \mathbb{G}_{a(1)} \\ 0 & 0 & 1 \end{pmatrix}.$$ □

We next discuss cohomology of abelian connected unipotent finite group schemes. Recall from Appendix A that as an augmented algebra, $kG$ is isomorphic to a tensor product of algebras of the form $kW_{m,1} = k[s]/(s^{p^m})$. Since cohomology of a finite group scheme $G$ in general only depends on the algebra structure of $kG$, not on the comultiplication, we get the following description of the cohomology ring.

**Theorem 9.2.** *Let $G$ be an abelian connected unipotent finite group scheme. The cohomology ring $H^*(G, k)$ is a tensor product of algebras of the form*

$$H^*(W_{m,1}, k) = k[x_m] \otimes \Lambda(\lambda_m)$$

*where $\lambda_m$ has degree one and $x_m$ has degree two.*
   *The surjective map $W_{m,1} \to W_{m-1,1}$ induces an inflation map*

$$H^*(W_{m-1,1}, k) \to H^*(W_{m,1}, k)$$

*sending $x_{m-1}$ to zero and $\lambda_{m-1}$ to $\lambda_m$. On the other hand, the injective map $W_{m-1,1} \to W_{m,1}$ induces a restriction map*

$$H^*(W_{m,1}, k) \to H^*(W_{m-1,1}, k)$$

*sending $x_m$ to $x_{m-1}$ and $\lambda_m$ to zero.*

*Proof.* The cohomology of the algebra $k[s]/(s^{p^m})$ and the restriction and inflation maps are well known from the cohomology theory of finite groups. See for example Chapter XII of Cartan and Eilenberg [17]. □

**Proposition 9.3.** *The cohomology of the supergroup scheme $W_{m,1}^-$ of Example 2.12 is given by*

$$H^{*,*}(W_{m,1}^-, k) = k[x_m, \zeta_m]/(\zeta_m^2)$$

*with $|x_m| = (2, 0)$ and $|\zeta_m| = (1, 1)$.*
   *For $m \geq 2$ the surjective map $W_{m,1}^- \to W_{m-1,1}^-$ induces an inflation map*

$$H^{*,*}(W_{m-1,1}^-, k) \to H^{*,*}(W_{m,1}^-, k)$$

*sending $x_{m-1}$ to zero and $\zeta_{m-1}$ to $\zeta_m$.*

*Proof.* The $E_2$ page of the spectral sequence

$$H^{*,*}(\mathbb{G}_a^-, H^{*,*}(W_{m,1}, k)) \Rightarrow H^{*,*}(G, k)$$

has a polynomial generator $\zeta_m$ on the base in degree $(1, 1)$, an exterior generator $\lambda_m$ on the fibre in degree $(1, 0)$ and a polynomial generator $x_m$ on the fibre in degree $(2, 0)$. The only differential is $d_2$, and this is determined by $d_2(\lambda_m) = \zeta_m^2$, $d_2(x_m) = 0$. The inflation maps follow from Theorem 9.2. □

**Proposition 9.4.** *If $G$ is a nonsplit extension*

$$1 \to \mathbb{G}_{a(r)} \to G \to \mathbb{G}_a^- \to 1$$

*with $r \geq 1$ then the inflation of $\zeta \in H^{1,1}(\mathbb{G}_a^-, k)$ to $G$ squares to zero in $H^{2,0}(G, k)$.*

26

*Proof.* By Corollary 2.8, we have a nonsplit extension

$$1 \to \mathbb{G}_{a(1)} \to G_{(1)} \to \mathbb{G}_a^- \to 1.$$

Hence, $G_{(1)} \cong W_{1,1}^-$ by Lemma 8.3. This implies that ignoring the comultiplication, we have $kG \cong k\mathbb{G}_{a(r-1)} \otimes kW_{1,1}^-$. The result follows from the case $m = 1$ of Proposition 9.3. □

**Lemma 9.5.** *If $G$ is an extension*

$$1 \to W_{2,2} \to G \to \mathbb{G}_a^- \to 1,$$

*then there exists a surjective map $G \to W_{2,1}$.*

*Proof.* Since $G = G_{(1)}G_{\mathsf{ev}}$ by Corollary 2.8, taking the first Frobenius kernels, we get an extension

$$1 \to W_{2,1} \to G_{(1)} \to \mathbb{G}_a^- \to 1.$$

Hence, $G/G_{(1)} \cong W_{2,2}/W_{2,1} \cong W_{2,1}$. □

**Lemma 9.6.** *Let $G$ be a unipotent finite supergroup scheme, and $f\colon G \to \overline{G} = \mathbb{G}_{a(r)} \times \mathbb{G}_a^- \times (\mathbb{Z}/p)^{\times s}$ a surjective map of supergroup schemes. Assume that*

    (a) *$f^*\colon H^{1,*}(\overline{G}, k) \to H^{1,*}(G, k)$ is an isomorphism, and*
    (b) *$f^*$ is one-to-one restricted to $H^2((\mathbb{Z}/p)^{\times s}, k) \subset H^{2,0}(\overline{G}, k)$*

*Then $\pi_0(G)$, the group of connected components of $G$, is isomorphic to $(\mathbb{Z}/p)^{\times s}$.*

*Proof.* Set $\pi = \pi_0(G)$ and let $\overline{\pi}$ be the Frattini quotient for $\pi$, that is, the maximal quotient isomorphic to an elementary abelian $p$-group. Then the map $f$ factors through $G^0 \rtimes \overline{\pi}$ and we have a commutative diagram

$$
\begin{array}{ccc}
G^0 \rtimes \pi & \longrightarrow & \pi \\
\downarrow & & \downarrow \\
G^0 \rtimes \overline{\pi} & \longrightarrow & \overline{\pi} \\
\downarrow & & \| \\
\overline{G} & \longrightarrow & (\mathbb{Z}/p)^{\times s}
\end{array}
$$

If $\pi \to \overline{\pi}$ is not an isomorphism, Lemma 3.5 implies that there exists an element $u$ in $H^2(\overline{\pi}, k) = H^2((\mathbb{Z}/p)^{\times s}, k)$ which pulls back to zero in $H^2(\pi, k)$ and, hence, in $H^{2,0}(G, k)$. Inflating the class $u$ to $H^{2,0}(\overline{G}, k)$, we get an element in $\mathrm{Ker}\, f^* \cap H^2((\mathbb{Z}/p)^{\times s}, k)$ contradicting assumption (b). Hence, $\pi \cong (\mathbb{Z}/p)^{\times s}$. □

The result below is a denouement of the preceding developments. It's import is that, in the situation of Theorem 4.3(ii), various finite (super)group schemes cannot be quotients of $G$, $G^0$ and $G_{\mathsf{ev}}^0$. Theorem 9.7 together with Theorem 4.4 are the major inputs in the proof of the detection Theorem 11.1.

**Theorem 9.7.** *Let $G$ be a unipotent finite supergroup scheme, and $f\colon G \to \overline{G} = \mathbb{G}_{a(r)} \times \mathbb{G}_a^- \times (\mathbb{Z}/p)^{\times s}$ a surjective map of supergroup schemes. Assume that*

    (a) *$f^*\colon H^{1,*}(\overline{G}, k) \to H^{1,*}(G, k)$ is an isomorphism, and*
    (b) *$I = \mathrm{Ker}\{f^*\colon H^{2,*}(\overline{G}, k) \to H^{2,*}(G, k)\}$ is one dimensional, spanned by an element of the form $\zeta^2 + \gamma x_r$ with $\gamma \neq 0 \in k$.*

*Then the following statements hold.*

    (I) *$G$ cannot have as a quotient the following supergroup schemes:*
        (i) *$H = (\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}) \rtimes (\mathbb{Z}/p)^{\times s}$,*
        (ii) *$H = (\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes (\mathbb{Z}/p)^{\times s}$.*
    (II) *The restriction $f_0 = f\!\downarrow_{G^0} : G^0 \to \overline{G}^0$ satisfies the following cohomological conditions:*
        ($a_0$) *$f_0^* : H^{1,*}(\overline{G}^0, k) \to H^{1,*}(G^0, k)$ is an isomorphism,*
        ($b_0$) *$\mathrm{Ker}\, f_0^* \cap H^{2,0}(\overline{G}^0, k)$ is one dimensional, spanned by $\zeta^2 + \gamma x_r$.*
    (III) *The following connected supergroup schemes $H$ cannot be quotients of $G^0$:*
        (i) *A nonsplit extension $1 \to \mathbb{G}_{a(r)} \to H \to \mathbb{G}_a^- \to 1$,*
        (ii) *$W_{m,1}^-$,*
        (iii) *$W_{2,1}$.*
    (IV) *$G_{\mathrm{ev}}^0$ cannot have $W_{2,2}$ as a quotient.*

*Proof.* (I). Let $\rho : G \to H$ be a surjective map of unipotent group schemes, and suppose that $H$ surjects further on a group scheme $H'$ which is isomorphic to $\mathbb{G}_{a(1)}$, $\mathbb{G}_a^-$ or $\mathbb{Z}/p$. By Remark 3.6, we have a commutative diagram

$$(9.1) \qquad\qquad \begin{array}{ccc} G & \xrightarrow{\;f\;} & \overline{G} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\overline{\rho}} \\ H & \xrightarrow{\;\chi\;} & H' \end{array}$$

Lemma 3.1 implies that $\overline{\rho} : \overline{G} \to H'$ induces an injective map on $H^{1,*}$. Moreover, the explicit calculation of cohomology for $\overline{G}$ further implies that the map $H^{*,*}(H', k) \to H^{*,*}(\overline{G}, k)$ is injective. Since $H' = \mathbb{G}_{a(1)}, \mathbb{G}_a^-$ or $\mathbb{Z}/p$, we have that $H^{1,*}(H', k)$ is a 1-dimensional vector space. Let $\alpha \in H^{1,*}(H', k)$ be a linear generator. Then the assumption $(a)$ together with the commutativity of (9.1) imply that

$$0 \neq (\overline{\rho} \circ f)^*(\alpha) = (\chi \circ \rho)^*(\alpha) \in H^{1,*}(G, k).$$

In Case (I.i), assume that there is a surjective map $G \to H$ where $H = (\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}) \rtimes (\mathbb{Z}/p)^{\times s}$. There are maps $\chi : H \to H'$ with $H' = \mathbb{G}_{a(1)}, \mathbb{Z}/p$. By Proposition 9.1, taking for $\alpha$ the elements $y$ and $\lambda$, we obtain a relation

$$f^*(\overline{\rho}^*(y)\overline{\rho}^*(\lambda)) = \rho^*(\chi^*(y)\chi^*(\lambda)) = 0.$$

Hence, $0 \neq \overline{\rho}^*(y)\overline{\rho}^*(\lambda)$ is in $I$ which contradicts the assumption $(b)$ completing the proof in that case.

In Case (I.ii), we assume there is a surjective map $G \to H$ where $H = (\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes (\mathbb{Z}/p)^{\times s}$. Cohomology of $H$ is computed explicitly in [14]; there exist non trivial elements $\lambda_1 \in H^{1,0}(H, k)$ and $\zeta \in H^{1,1}(H, k)$ such that $\lambda_1 \zeta = 0$. Arguing as in (I.i), we get a contradiction with the assumption $(b)$ again.

(II.$a_0$). Let $\pi = \pi_0(G)$ be the group of connected components of $G$. By Lemma 9.6, we have $\pi \cong (\mathbb{Z}/p)^{\times s}$, which is the same as $\overline{\pi} = \pi_0(\overline{G})$. The map $f : G \to \overline{G}$ induces a

commutative diagram of 5 term sequences:

$$
\begin{array}{ccccccccc}
H^{1,*}(\pi, k) & \longrightarrow & H^{1,*}(G, k) & \longrightarrow & H^{1,*}(G^0, k)^\pi & \longrightarrow & H^{2,*}(\pi, k) & \longrightarrow \\
\big\| & & \big\uparrow{\scriptstyle f^*} & & \big\uparrow{\scriptstyle f_0^*} & & \big\| & \\
H^{1,*}(\overline{\pi}, k) & \longrightarrow & H^{1,*}(\overline{G}, k) & \longrightarrow & H^{1,*}(\overline{G}^0, k) & \xrightarrow{\ 0\ } & H^{2,*}(\overline{\pi}, k) & \longrightarrow
\end{array}
$$

Since $f^*$ is an iso on $H^{1,*}$, we conclude that it induces an isomorphism $H^{1,*}(\overline{G}^0, k) \cong H^{1,*}(G^0, k)^\pi$. It remains to show that $\pi$ acts trivially on $H^{1,*}(G^0, k)$.

By Lemma 3.1, We have $H^{1,*}(G^0, k) \cong H^{1,0}(G^0, k) \oplus H^{1,1}(G^0, k)$ with $H^{1,0}(G^0, k) \cong \mathrm{Hom}(G^0, \mathbb{G}_a)$, $H^{1,1}(G^0, k) \cong \mathrm{Hom}(G^0, \mathbb{G}_a^-)$ by Lemma 3.1, and the action of $\pi$ fixing the even and odd parts.

The assumption that $f^*$ is an isomorphism on $H^1$ together with Lemma 3.1 imply that

(9.2)
$$
\dim \mathrm{Hom}(G^0, \mathbb{G}_a^-)^\pi = 1
$$

(9.3)
$$
\mathrm{Hom}(G^0, \mathbb{G}_a)^\pi = \mathrm{Hom}(\mathbb{G}_{a(r)}, \mathbb{G}_{a(r)}).
$$

Hence, to show that $\pi$ acts trivially on $H^{1,*}(G_0, k)$, we need to show the same two equalities for $\mathrm{Hom}(G^0, \mathbb{G}_a^-)$ and $\mathrm{Hom}(G^0, \mathbb{G}_a)$.

We first show that $\dim_k \mathrm{Hom}(G^0, \mathbb{G}_a^-) = 1$. Suppose $\dim_k \mathrm{Hom}(G^0, \mathbb{G}_a^-) \geq 2$. Since $\pi$ is a $p$-group, there exists a two-dimensional $\pi$-invariant subspace of $\mathrm{Hom}(G^0, \mathbb{G}_a^-)$ and, hence, a $\pi$-invariant quotient of the form $\mathbb{G}_a^- \times \mathbb{G}_a^-$. But this implies that $G$ has a quotient of the form $H = (\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes (\mathbb{Z}/p)^{\times s}$ which is disallowed by (I.ii). Hence, $\dim_k \mathrm{Hom}(G^0, \mathbb{G}_a^-) = 1$.

We now consider $\mathrm{Hom}(G^0, \mathbb{G}_a)$. First, since $G^0$ is finite, there exists $n$ such that $\mathrm{Hom}(G^0, \mathbb{G}_a) = \mathrm{Hom}(G^0, \mathbb{G}_{a(n)})$. Pick teh maximal $n$ so that the map $G^0 \to \mathbb{G}_{a(n)}$ is surjective. The standard projection $\mathbb{G}_{a(n)} \to \mathbb{G}_{a(1)}$ induces a map on Hom spaces $\mathrm{Hom}(G^0, \mathbb{G}_{a(n)}) \to \mathrm{Hom}(G^0, \mathbb{G}_{a(1)})$; the action of $\pi$ descends along this map since the Frobenius map is $\pi$-equivariant. If $\dim_k \mathrm{Hom}(G^0, \mathbb{G}_{a(1)}) > 1$, then arguing just as in the case of $\mathbb{G}_a^-$ we deduce a contradiction with (I.i). Hence, $\mathrm{Hom}(G^0, \mathbb{G}_{a(1)}) = 1$. Therefore,

$$
\mathrm{Hom}(G^0, \mathbb{G}_{a(n)}) \cong \mathrm{Hom}(\mathbb{G}_{a(n)}, \mathbb{G}_{a(n)})
$$

It remains to show that $n = r$. Note that $\mathrm{Hom}(\mathbb{G}_{a(n)}, \mathbb{G}_{a(n)}) \simeq \mathbb{G}_a^{\times n}$ as a group scheme, with the action of $\pi$ preserving the group scheme structure. Since $\mathbb{G}_a^{\times n}$ is connected, the action of $\pi$ must be trivial, hence, $r = n$.

(II.$b_0$). The projection $f \colon G \to \overline{G}$ induces a map on spectral sequences making the following diagram commute:

(9.4)
$$
\begin{array}{ccc}
H^{*,*}(\pi, H^{*,*}(G^0, k)) & \Longrightarrow & H^{*,*}(G, k) \\
\big\uparrow & & {\scriptstyle f^*}\big\uparrow \\
H^{*,*}(\overline{\pi}, H^{*,*}(\overline{G}^0, k)) = H^{*,*}(\mathbb{Z}/p^{\times s}, k) \otimes H^{*,*}(\overline{G}^0, k) & \xrightarrow{\ \sim\ } & H^{*,*}(\overline{G}, k)
\end{array}
$$

Here, the star for the internal degree is preserved by the spectral sequence. Since $\zeta^2 + \gamma x_r \in H^{2,0}(\overline{G}^0, k) = H^{2,0}(\mathbb{G}_{a(r)} \times \mathbb{G}_a^-, k)$, we conclude that it belongs to the kernel of $f_0^*$. It remains to show that this class generates the kernel of $f_0^*$ on $H^{2,0}$.

29

Let

$$(9.5) \qquad H^{2,0}(G,k) \longleftarrow F^1 H^{2,0}(G,k) \longleftarrow F^0 H^{2,0}(G,k)$$

$$\begin{array}{ccc} & \Big\downarrow \equiv & \Big\downarrow \equiv \\ & F^1 H^{2,0}(\overline{G},k) \longleftarrow F^0 H^{2,0}(\overline{G},k) \end{array}$$

be the filtration on $H^{2,0}$ with subquotients giving the $E_\infty$ term of the spectral sequences.

We consider another diagram induced by $f$:

$$(9.6) \qquad \begin{array}{ccc} H^{2,0}(\overline{G}^0,k) & \xrightarrow{f_0^*} & H^{2,0}(G^0,k)^\pi \\ \Big\uparrow \Big\uparrow{\rho} & & \Big\uparrow{i} \\ H^{2,0}(\overline{G},k) & \xrightarrow{f^*} & H^{2,0}(G,k) \end{array}$$

The left vertical map induced by the embedding $\overline{G}^0 < \overline{G}$ splits since

$$H^{2,0}(\overline{G},k) \cong H^2(\pi,k) \oplus H^1(\pi, H^{1,0}(\overline{G}^0,k)) \oplus H^{2,0}(\overline{G}^0,k).$$

The left vertical map $\rho \colon H^{2,0}(\overline{G}^0,k) \hookrightarrow H^{2,0}(\overline{G},k)$ is the identification of $H^{2,0}(\overline{G}^0,k)$ with the last direct summand.

The right vertical map $i \colon H^{2,0}(G,k) \to H^{2,0}(G^0,k)^\pi$ is the edge homomorphism of the top row spectral sequence in (9.4), hence,

$$(9.7) \qquad \operatorname{Ker} i = F^1 H^{2,0}(G,k).$$

Let $\alpha \in H^{2,0}(\overline{G}^0,k)$ be a class in the kernel of $f_0^*$. Then $f_0^*(\alpha) = i f^* \rho(\alpha) = 0$ implies that $f^* \rho(\alpha) \in \operatorname{Ker} i = F^1 H^{2,0}(G,k)$. Since $F^1 H^{2,0}(G,k) \cong F^1 H^{2,0}(\overline{G},k)$ by (9.5), there exists $\beta \in F^1 H^{2,0}(\overline{G},k) = H^2(\pi,k) \oplus H^1(\pi, H^{1,0}(\overline{G}^0,k))$, such that $f^*(\rho(\alpha)) = f^*(\beta)$, that is,

$$f^*(\rho(\alpha) - \beta) = 0.$$

Assumption (b) now implies that $\rho(\alpha) - \beta$ is a multiple of $\zeta^2 + \gamma x_r$ and, hence, $\rho(\alpha) - \beta \in \operatorname{Im} \rho$. Therefore, $\beta \in \operatorname{Im} \rho$. This implies that $\beta = 0$ since $\operatorname{Im} \rho \cap F^1 H^{2,0}(\overline{G},k) = 0$. We conclude that $f^*(\rho(\alpha)) = 0$, and, hence, $\alpha$ is a multiple of $\zeta^2 + \gamma x_r$. Hence the kernel is one-dimensional.

(III). We apply the same argument as in Case (I) but to $f_0 \colon G^0 \to \overline{G}^0$. Once again, we have a commutative diagram of surjective maps:

$$\begin{array}{ccc} G^0 & \xrightarrow{f_0} & \overline{G}^0 \\ \Big\downarrow{\rho} & & \Big\downarrow{\overline{\rho}} \\ H & \xrightarrow{\chi} & H' \end{array}$$

For (III.i), Proposition 9.4 gives an element $\zeta \in H^{1,1}(H',k)$ such that $\chi^*(\zeta)^2 = 0$. Hence, commutativity of the diagram above implies that $0 \neq (\overline{\rho}^*(\zeta))^2$ is in the kernel of $f_0^*$ contradicting the assumption II($b_0$), and completing the proof in this case.

Case (III.ii) follows from Proposition 9.3 in a similar fashion taking $H' = \mathbb{G}_a^1$ and $\alpha = \zeta_m$.

If $G^0$ has a quotient $W_{2,1}$, then $\beta \mathscr{P}^0(\lambda_2)$, where $\lambda_2$ is a degree $(1,0)$ cohomology generator of $H^{*,*}(W_{2,1},k)$ is in the kernel of $f_0^*$, contradicting II($b_0$).

Finally, Case (IV) follows from Lemma 9.5 and case (II.iii). $\qquad\qquad \square$

**Corollary 9.8.** *Let $G$ be a unipotent finite supergroup scheme satisfying the assumptions of Theorem 9.7. Let $A = G/[G_{\text{ev}}, G_{\text{ev}}]$. Then $A^0_{\text{ev}}$ is isomorphic to a quotient of $E_{m,n} = (E^-_{m,n})_{\text{ev}}$ for some $m, n > 0$.*

*Proof.* First we claim that $\dim_k \text{Hom}_{\text{Gr}/k}(A_{\text{ev}}, \mathbb{G}_{a(1)}) = 1$. This is because if this dimension is two or greater then $G$, and, hence, $G^0$, has a quotient which is a nonsplit extension of $\mathbb{G}^-_a$ by $\mathbb{G}_{a(1)}$, which is not allowed by Theorem 9.7.

Next, we claim that $\dim_k \text{Hom}_{\text{Gr}/k}(A^0_{\text{ev}}, \mathbb{G}_{a(1)}) = 1$. This is because if this dimension is two or greater then $G$ has a quotient which is a semidirect product $(\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}) \rtimes (\mathbb{Z}/p)^{\times s}$ with non-trivial action. This is once again disallowed by Theorem 9.7.

By Theorem 9.7(III), $A^0_{\text{ev}}$ does not have $W_{2,2}$ as a quotient. Together with the condition $\dim_k \text{Hom}_{\text{Gr}/k}(A^0_{\text{ev}}, \mathbb{G}_{a(1)}) = 1$ this allows us to apply Lemma A.2, concluding that $A^0_{\text{ev}}$ is isomorphic to a quotient of the group scheme $E_{m,n}$. $\square$

Now for the promised computation of cohomology of Witt elementary supergroup schemes.

**Theorem 9.9.** *The cohomology of the group $E^-_{m,n}$ (as defined in (8.5)) ($m \geq 2$, $n \geq 1$) is given by*

$$H^{*,*}(E^-_{m,n}, k) = k[x_{m,1}, \ldots, x_{m,n}, \zeta_m] \otimes \Lambda(\lambda_{m,1}, \ldots, \lambda_{m,n})$$

*with $|x_{m,i}| = (2,0)$, $|\zeta_m| = (1,1)$ and $|\lambda_{m,i}| = (1,0)$.*

*For $m \geq 3$, the surjective map $E^-_{m,n} \to E^-_{m-1,n}$ induces an inflation map*

$$H^{*,*}(E^-_{m-1,n}, k) \to H^{*,*}(E^-_{m,n}, k)$$

*sending $x_{m-1,i}$ to $x_{m,i}$ $(1 \leq i \leq n-1)$, $x_{m-1,n}$ to zero, $\zeta_{m-1}$ to $\zeta_m$ and $\lambda_{m-1,i}$ to $\lambda_{m,i}$ $(1 \leq i \leq n)$.*

*The surjective map $E^-_{2,n} \to E^-_{1,n} = \mathbb{G}_{a(n)} \times \mathbb{G}^-_a$ induces an inflation map sending $x_i$ to $x_{2,i}$ $(1 \leq i \leq n-1)$, $x_n$ to $\zeta^2$, $\zeta_2$ to $\zeta$ and $\lambda_i$ to $\lambda_{2,i}$. In particular, the kernel of*

$$H^{2,0}(\mathbb{G}_{a(n)} \times \mathbb{G}^-_a, k) \to H^{2,0}(E^-_{2,n}, k)$$

*is one dimensional, spanned by $\zeta^2 - x_n$.*

*Proof.* Again, we use the fact that the cohomology only depends on the algebra structure of the group algebra and not on the comultplication. The algebra structure is described in Definition 8.6, and is a tensor product $k\mathbb{G}_{a(n-1)} \otimes kE^-_{m,1}$. The first factor gives the generators $\lambda_{m,1}, \ldots, \lambda_{m,n-1}, x_{m,1} \ldots, x_{m,n-1}$, so we need to compute $H^{*,*}(E^-_{m,1}, k)$. We do this using the spectral sequence

$$H^{*,*}(\mathbb{G}^-_a, H^{*,*}(W_{m,1}, k)) \Rightarrow H^{*,*}(E^-_{m,1}, k).$$

This has the same $E_2$ page as the spectral sequence in the proof of Proposition 9.3, but all the differentials are zero. This accounts for the generators $x_{m,n}$, $\zeta_m$ and $\lambda_{m,n}$. The inflation maps again follow from Theorem 9.2. $\square$

**Theorem 9.10.** *The cohomology of the group $E^-_{m,n,\mu}$ of (8.10) is given by*

$$H^{*,*}(E^-_{m,n,\mu}, k) = k[x_{m,1,\mu}, \ldots, x_{m,n,\mu}, \zeta_{m,\mu}] \otimes \Lambda(\lambda_{m,1,\mu}, \ldots, \lambda_{m,n,\mu})$$

*with $|x_{m,i,\mu}| = (2,0)$, $|\zeta_{m,\mu}| = (1,1)$ and $|\lambda_{m,i,\mu}| = (1,0)$.*

*The surjective map $E^-_{m+1,n+1} \to E^-_{m,n,\mu}$ induces an inflation map*

$$H^{*,*}(E^-_{m,n,\mu}, k) \to H^{*,*}(E^-_{m,n}, k)$$

*sending each element to the corresponding element without the subscript $\mu$, except that it sends $x_{m,n,\mu}$ to zero.*

*Proof.* The proof is essentially the same as for Theorem 9.9. □

*Remark* 9.11. The computation in Theorem 9.9 also appears in Proposition 3.2.1 (1) and (3) and Lemma 3.2.4 of Drupieski and Kujawa [24]. Similarly, Theorem 9.10 should be compared with Proposition 3.2.1 (4) and (5) of [24] and Lemma 3.1.1 (3) and Remark 2.2.3 (1) of [25].

*Remark* 9.12. We tabulate the action of the Steenrod operations on $H^{*,*}(E^-_{m,n}, k)$, for use in the proof of Theorem 10.3. The table for $H^{*,*}(E^-_{m,n,\mu}, k)$ looks exactly the same after adding $\mu$ to all the indices; cf. Table 1.

| degree | | $\mathscr{P}^0$ | $\beta\mathscr{P}^0$ | $\mathscr{P}^{\frac{1}{2}}$ | $\beta\mathscr{P}^{\frac{1}{2}}$ | $\mathscr{P}^1$ | $\mathscr{P}^i$ $(i \geq 2)$ | $\beta\mathscr{P}^i$ $(i \geq 1)$ | |
|---|---|---|---|---|---|---|---|---|---|
| $\lambda_{m,i}$ | $(1,0)$ | $\lambda_{m,i+1}$ | $-x_{m,i}$ | | | $0$ | $0$ | $0$ | $1 \leq i < n$ |
| $\lambda_{m,n}$ | $(1,0)$ | $0$ | $-\zeta_m^2$ | | | $0$ | $0$ | $0$ | |
| $\zeta_m$ | $(1,1)$ | | | $\zeta_m^p$ | $0$ | | | | |
| $x_{m,i}$ | $(2,0)$ | $x_{m,i+1}$ | $0$ | | | $x_i^p$ | $0$ | $0$ | $1 \leq i < n$ |
| $x_{m,n}$ | $(2,0)$ | $0$ | $0$ | | | $0$ | $0$ | $0$ | |

## 10. Cohomological characterisation of elementary supergroups

The purpose of this section is to show that elementary supergroups as defined in 1.1 can be characterised cohomologically. Recall that for $\overline{G} = \mathbb{G}_{a(r)} \times \mathbb{G}_a^- \times (\mathbb{Z}/p)^{\times s}$, we employ the following notation for the standard generators in cohomology:

$$H^{*,*}(\overline{G}, k) = k[x_1, \ldots, x_r] \otimes \Lambda(\lambda_1, \ldots, \lambda_r) \otimes k[\zeta] \otimes k[z_1, \ldots, z_s] \otimes \Lambda(y_1, \ldots, y_s)$$

Theorems 9.9 and 9.10 show that if $G$ is an elementary supergroup scheme equipped with a surjection $G \to \overline{G}$ which induces an isomorphism on $H^{1,*}$, then either $f$ is an isomorphism or $\mathrm{Ker}\, f^*$ falls under the case (ii) of Theorem 4.3. Theorem 10.3 proves a partial converse to this statement, and is the key step in the proof of Theorem 1.2.

**Lemma 10.1.** *Let $1 \to Z \xrightarrow{f} G \xrightarrow{\psi} A \to 1$ be a central extension of group schemes with $Z \cong \mathbb{G}_{a(1)}$ and $A$ abelian. If the connecting homomorphism $d_2 \colon H^1(Z, k) \to H^2(A, k)$ is zero then $G$ is abelian.*

*Proof.* The five term sequence of the central extension shows that there is an element $\tilde{u} \in H^1(G, k)$ whose restriction is $f^*(\tilde{u}) = u \in H^1(Z, k)$. Applying Lemma 3.1, we see that there is a homomorphism $\phi \colon G \to \mathbb{G}_a$ whose composite with $Z \to G$ is nonzero. Then $(\psi, \phi) \colon G \to A \times \mathbb{G}_a$ is an embedding, and $G$ is a subgroup scheme of an abelian group scheme, hence abelian. □

The following proposition, which is the key observation necessary for the proof of Theorem 10.3, gives a cohomological criterion to establish that certain extensions of abelian finite group schemes are abelian themselves.

**Proposition 10.2.** *Let $1 \to Z \to G \to A \to 1$ be a central extension of group schemes with $Z \cong \mathbb{G}_{a(1)}$ and $A$ abelian. The following are equivalent:*

(i) *$G$ is abelian.*

(ii) *There exists an abelian finite group scheme $A'$ and a surjective map $A' \to A$ such that the composition $H^1(Z, k) \xrightarrow{d_2} H^2(A, k) \longrightarrow H^2(A', k)$ is zero. the induced map in cohomology sends $d_2(\lambda) \in H^2(A, k)$ to zero in $H^2(A', k)$ for all $0 \neq \lambda \in H^1(Z, k)$.*

*Proof.* (i) $\Rightarrow$ (ii): Take $A' = G$ and use the five term sequence.

(ii) $\Rightarrow$ (i): Let $1 \to A'' \to A' \to A \to 1$ be the short exact sequence given by the surjection $A' \to A$. Form the pullback $X$ of $G \to A$ and $A' \to A$:

$$
\begin{array}{ccc}
1 & & 1 \\
\downarrow & & \downarrow \\
Z & == & Z \\
\downarrow & & \downarrow \\
1 \longrightarrow A'' \longrightarrow X \longrightarrow G \longrightarrow 1 \\
\| \quad\quad \downarrow \quad\quad \downarrow \\
1 \longrightarrow A'' \longrightarrow A' \longrightarrow A \longrightarrow 1 \\
\downarrow \quad\quad \downarrow \\
1 \quad\quad 1
\end{array}
$$

If $d_2(\lambda)$ goes to zero in $H^2(A', k)$ then the sequence

$$1 \to Z \to X \to A' \to 1$$

satisfies the conditions of Lemma 10.1, and so $X$ is abelian. Since $G$ is a quotient of $X$, it follows that $G$ is abelian. $\qquad\square$

**Theorem 10.3.** *Let $G$ be a unipotent finite supergroup scheme, and $f \colon G \to \overline{G} = \mathbb{G}_{a(r)} \times \mathbb{G}_a^- \times (\mathbb{Z}/p)^{\times s}$ a surjective map of supergroup schemes. Assume that*

(1) *$f^* \colon H^{1,*}(\overline{G}, k) \to H^{1,*}(G, k)$ is an isomorphism,*
(2) *$\mathrm{Ker}(f^*) \cap H^{2,0}(\overline{G}, k)$ is one dimensional, spanned by an element of the form $\zeta^2 + \gamma x_r$ with $0 \neq \gamma \in k$.*
(3) *$\mathrm{Ker}(f^*) \cap H^{2,1}(\overline{G}, k) = 0$,*
(4) *There does not exist $i \in \mathbb{Z}_{\geq 0}$ and $y \in H^1((\mathbb{Z}/p)^{\times s}, \mathbb{F}_p) \subset H^{1,0}(\overline{G}, k)$ such that $\zeta^{2p^i} \beta \mathscr{P}^0(y)$ or $\zeta^{2p^i+2}$ lie in $\mathrm{Ker}\{f^* \colon H^{*,*}(\overline{G}, k) \to H^{*,*}(G, k)\}$.*

*Then $G$ is isomorphic to $E_{m,r}^- \times (\mathbb{Z}/p)^{\times s}$ or $E_{m,r+1,\mu}^-$ for some $m \geq 1$, $\mu \in k$.*

*Proof.* The proof has three essential reduction steps:

Step (1) The first step is to show that $G_{\mathrm{ev}}$ is normal in $G$, and $G/G_{\mathrm{ev}} \cong \mathbb{G}_a^-$.
Step (2) Let $A = G/[G_{\mathrm{ev}}, G_{\mathrm{ev}}]$. The second step is to show that $A$ is isomorphic to either $E_{m,r}^- \times (\mathbb{Z}/p)^{\times s}$ or $E_{m,r+1,\mu}^-$ for some $m \geq 1$, $\mu \in k$.
Step (3) Finally, we show that $G \cong A$.

By Lemma 9.6, $\pi \cong (\mathbb{Z}/p)^{\times s}$.

Let $\psi \colon \overline{G} \to \mathbb{G}_a^-$ be the projection map, and let $H = \mathrm{Ker}\{\psi \circ f \colon G \to \overline{G} \to \mathbb{G}_a^-\}$. We now show that $H = G_{\mathrm{ev}}$, proving Step (1).

We have the five-term sequence associated with the extension $1 \to H \to G \to \mathbb{G}_a^- \to 1$ of which we only need the odd internal degree part:

$$0 \longrightarrow H^{1,1}(\mathbb{G}_a^-, k) \xrightarrow[(\psi \circ f)^*]{\sim} H^{1,1}(G, k) \xrightarrow[=0]{\mathrm{res}} H^{1,1}(H, k)^{\mathbb{G}_a^-} \xrightarrow[=0]{d_2} H^{2,1}(\mathbb{G}_a^-, k) \xhookrightarrow{f^*} H^{2,1}(G, k).$$

The first map is an isomorphism since $f^*$ is an isomorphism by assumption (i), and $\psi^*$ is an isomorphism on $H^{1,1}$ since we know cohomology of $\overline{G}$ and $\mathbb{G}_a^-$ explicitly. Assumption (iii) implies that the last map is an embedding. Hence, $H^{1,1}(H, k)^{\mathbb{G}_a^-} = 0$ and, therefore, $H^{1,1}(H, k) = 0$ since $\mathbb{G}_a^-$ is unipotent. We conclude that $H$ is even by Lemma 3.4. Since $G/H \cong \mathbb{G}_a^-$, $H$ is the largest even subgroup scheme; hence, $H = G_{\mathsf{ev}}$ which proves the claim.

Assumption (1) implies that $r$ is maximal such that there is a surjective $\pi$-invariant map $G^0 \to \mathbb{G}_{a(r)}$ since any $\pi$-invariant surjection $G^0 \to \mathbb{G}_{a(s)}$ induces an embedding in cohomology $H^1(\mathbb{G}_{a(s)}, k) \hookrightarrow H^{1,0}(G, k)$ by Lemma 3.1. We claim that $r$ is also maximal subject to the existence of a $\pi$-invariant surjective map $G_{\mathsf{ev}}^0 \to \mathbb{G}_{a(r)}$. Suppose, to the contrary, that there is a $\pi$-invariant surjective map $G_{\mathsf{ev}}^0 \to \mathbb{G}_{a(r+1)}$, and let $N$ be the kernel. Since $G_{\mathsf{ev}}^0 = G^0 \cap G_{\mathsf{ev}}$, we have that $G^0/G_{\mathsf{ev}}^0 \cong G/G_{\mathsf{ev}} \cong \mathbb{G}_a^-$. We have a commutative diagram of $\pi$-invariant homomorphisms:

$$
\begin{array}{ccccccccc}
& & N & = \!\!= \!\!= \!\!= \!\!= & N & & & & \\
& & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & G_{\mathsf{ev}}^0 & \longrightarrow & G^0 & \longrightarrow & \mathbb{G}_a^- & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbb{G}_{a(r+1)} & \longrightarrow & \overline{G}^0 = G^0/N & \longrightarrow & \mathbb{G}_a^- & \longrightarrow & 1
\end{array}
$$

If the extension on the bottom row splits, then it $\pi$-splits by Lemma 8.9. Hence, there is a $\pi$-invariant surjective map $G^0 \to \mathbb{G}_{a(r+1)}$ which contradicts maximality of $r$. On the other hand, if the map does not split, then the inflation of $\zeta \in H^{1,1}(\mathbb{G}_a^-, k)$ to $\overline{G}^0$ is a non trivial cohomology class in $H^{1,1}(\overline{G}^0, k)$ which squares to zero in $H^{2,0}(G, k)$ by Proposition 9.4. Inflating $\zeta$ further to $G^0$ via the projection $G^0 \to \overline{G}^0$, we get a non trivial $\pi$-invariant cohomology class in $H^{1,1}(G^0, k)$ which squares to 0. Hence $\zeta^2$ is in the kernel of the map $f^* : H^{2,0}(\overline{G}, k) \to H^{2,0}(G, k)$ which contradicts assumption (2). We therefore conclude that $r$ is maximal such that there is a surjective map $G_{\mathsf{ev}}^0 \to \mathbb{G}_{a(r)}$ as claimed.

Since $G_{\mathsf{ev}} \trianglelefteq G$ is a normal subgroup scheme, Lemma 2.18 implies that $[G_{\mathsf{ev}}, G_{\mathsf{ev}}] \trianglelefteq G$. Let $A = G/[G_{\mathsf{ev}}, G_{\mathsf{ev}}]$, so that $A_{\mathsf{ev}}$ is the abelianisation of $G_{\mathsf{ev}}$.

*Claim* 10.3.1. We have that $A$ is isomorphic to $E_{m,r}^- \times (\mathbb{Z}/p)^{\times s}$ or $E_{m,r+1,\mu}^- \times (\mathbb{Z}/p)^{\times s}$ for some $m \geq 1$.

*Proof of the Claim.* Corollary 9.8 implies that $A_{\mathsf{ev}}^0$ is isomorphic to a quotient of $E_{m,n}$ for some $m, n \geq 1$. By Theorem 8.13, this implies that $A_{\mathsf{ev}}^0$ is isomorphic either to $E_{m,n}$ or to $E_{m,n,\mu}$ for some $0 \neq \mu \in k$. We divide into two cases according to these two possibilities. Looking at homomorphisms from these to $\mathbb{G}_{a(r)}$, we see that in the first case $n = r$, while in the second case $n = r + 1$.

*Case I:* $A_{\mathsf{ev}}^0 \cong E_{m,r}$. This case splits further into two subcases.

(1) $r = 1$. Since $E_{m,1} = W_{m,1}$, we have that $A^0 = A^0_{(1)}$ fits in the extension

$$1 \to W_{m,1} \to A^0 \to \mathbb{G}_a^- \to 1,$$

and, hence, is described by Lemma 8.4. The cohomological restriction in the assumption (2) implies that the only allowed possibility is $A^0 \cong E_{m,1}^-$ since in any other case $A^0$, and, hence, $G^0$, will have a quotient isomorphic to $W_{2,1}$ or $W_{1,1}^-$ which are disallowed by Theorem 9.7.III(iii). Hence, $A \cong E_{m,1}^- \times (\mathbb{Z}/p)^{\times s}$. In terms of the diagrams for the possibilities for $A^0$ as in Lemma 8.4, the only way to attach the node $\sigma$ to avoid quotients isomorphic to $W_{2,1}$ and $W_{1,1}^-$ is to the node marked with $s^{[p]}$.

(2) $r > 1$. In this case, $kA^0$ is described by Lemma 8.11(1). The coefficient $\alpha$ in the relation $\sigma^2 - s^{[p]^i} - \alpha s_1$ must be zero since $G^0$, and, hence, $A^0$, has a quotient isomorphic to $\mathbb{G}_{a(r)}$. The parameter $i$ must be 1 since for $i > 1$ there will be a quotient isomorphic to $W_{2,1}$. In terms of the picture in Lemma 8.11, the node $\sigma$ can be connected *only* to the node $s^{[p]}$, for otherwise the top two nodes on the left arm will form a quotient isomorphic $W_{2,1}$. Hence, $A^0 \cong E_{m,r}^-$. Since $A_{\mathrm{ev}}$ is abelian, the group of connected components of $A$ acts trivially on $A^0_{\mathrm{ev}}$. Since $(\mathbb{Z}/p)^{\times s}$ is a $p$-group, it also acts trivially on the quotient $A^0/A^0_{\mathrm{ev}} \cong \mathbb{G}_a^-$. Therefore, $A \cong E_{m,r}^- \times (\mathbb{Z}/p)^{\times s}$.

*Case II:* $A^0_{\mathrm{ev}} \cong E_{m,r+1,\mu}$. This case is similar. The possibilities for $kA^0$ are given by Lemma 8.11(ii). All of them but one are disallowed by Theorem 9.7.III(iii). We conclude that $A^0 \cong E_{m,r+1,\mu}^-$, and, therefore, we can identify $A$ with $E_{m,r+1,\mu}^- \times (\mathbb{Z}/p)^{\times s}$. □

Now that we have identified $A = G/[G_{\mathrm{ev}}, G_{\mathrm{ev}}]$, it remains to show that $G = A$, that is $[G_{\mathrm{ev}}, G_{\mathrm{ev}}] = 1$. We prove this by contradiction. Assume that $G \neq A$.

Note that $[G_{\mathrm{ev}}, G_{\mathrm{ev}}] \subseteq [G, G] \subseteq G^0$ since the group of connected components of $G$ is abelian. Hence, $[G_{\mathrm{ev}}, G_{\mathrm{ev}}]$ is a connected unipotent finite group scheme. Therefore, there exists a maximal proper subgroup $N$ of $[G_{\mathrm{ev}}, G_{\mathrm{ev}}]$ such that $[G_{\mathrm{ev}}, G_{\mathrm{ev}}]/N \cong \mathbb{G}_{a(1)}$ giving rise to a central extension

(10.1) $$1 \to \mathbb{G}_{a(1)} \to G/N \to A \to 1.$$

Let $\psi \colon G \to A$, $\phi \colon A \to \overline{G}$ be the projection maps; we factor $\psi$ as $G \xrightarrow{\psi_2} G/N \xrightarrow{\psi_1} A$. The map $f \colon G \to \overline{G}$ then factors as follows:

$$f \colon G \xrightarrow{\psi_2} G/N \xrightarrow{\psi_1} A \xrightarrow{\phi} \overline{G} = \mathbb{G}_{a(r)} \times \mathbb{G}_a^- \times (\mathbb{Z}/p)^{\times s}$$

Since $\phi$, $\psi$ are surjective, the induced maps on $H^1$ are injective. Since the composition

$$f \colon H^{1,*}(\overline{G}, k) \xhookrightarrow{\phi^*} H^{1,*}(A, k) \xhookrightarrow{\psi^*} H^{1,*}(G, k)$$

is an isomorphism, we conclude that

$$\psi^* \colon H^{1,*}(A, k) \xrightarrow{\sim} H^{1,*}(G, k)$$

is also an isomorphism.

We again consider two cases: $A \cong E_{m,r}^- \times (\mathbb{Z}/p)^{\times s}$ and $A \cong E_{m,r+1,\mu}^- \times (\mathbb{Z}/p)^{\times s}$.

*Case I:* $A \cong E_{m,r}^- \times (\mathbb{Z}/p)^{\times s}$. Assume that $m \geq 2$. By Theorem 9.9,

(1) $H^{*,*}(A, k) \cong k[x_{m,1}, \ldots, x_{m,n}, \zeta_m] \otimes \Lambda(\lambda_{m,1}, \ldots, \lambda_{m,n}) \otimes k[z_1, \ldots, z_s] \otimes \Lambda(y_1, \ldots, y_s)$,

(2) $\phi^* \colon H^{1,*}(\overline{G}, k) \to H^{1,*}(A, k)$ is an isomorphism,

35

(3) $\operatorname{Ker}\{\phi^* : H^{2,*}(\overline{G}, k) \to H^{2,*}(A, k)\} = k\langle \zeta^2 + \gamma x_r \rangle$.

In particular, $\phi^*$ and $f^*$ have the same kernel on $H^{2,*}(\overline{G}, k)$. Therefore, $\psi_1^* \colon H^{2,*}(A, k) \to H^{2,*}(G/N, k)$ is one-to-one restricted to $\phi^*(H^{2,*}(\overline{G}, k))$.

Consider the five term exact sequence induced by (10.1):

$$H^{1,*}(A, k) \longrightarrow H^{1,*}(G/N, k) \longrightarrow H^{1,*}(\mathbb{G}_{a(1)}, k) \xrightarrow{d_2} H^{2,*}(A, k) \xrightarrow{\psi_1^*} H^{2,*}(G/N, k)$$

Let $\lambda \in H^{1,0}(\mathbb{G}_{a(1)}, k)$ be any linear generator. Since $G_{\mathsf{ev}}/N$ is non–abelian, $d_2(\lambda) \in H^{2,0}(A, k)$ is a nonzero element in the kernel of $\psi_1^*$ by Lemma 10.1, and, hence, a non–zero element in the kernel of $\psi^*$. By Theorem 9.9, the only linear generator of $H^{2,0}(A, k)$ which is *not* in the image of $\phi^*$, is $x_{m,r}$. Hence, replacing $\lambda$ by a non–zero multiple if necessary, we may assume that

$$d_2(\lambda) = x_{m,r} + u$$

where $u \in \phi^*(H^{2,*}(\overline{G}, k))$.

*Claim* 10.3.2. $u = \alpha \zeta^2$ for some $\alpha \in k$.

*Proof.* We prove the claim by consecutive application of Steenrod operations, similarly to Theorem 4.3. Since any element in $\operatorname{Ker}\{\psi^* : H^{2,*}(A, k) \to H^{2,*}(G, k)\}$ must have the form $a x_{m,r} + v$ with $a \neq 0$ and $v \in \phi^*(H^{2,0}(\overline{G}, k))$, we have

$$\dim_k \operatorname{Ker}\{\psi^* \colon H^{2,*}(A, k) \to H^{2,*}(G, k)\} = 1$$

Let

$$u = \alpha \zeta_m^2 + \sum_{1 \leq i < j \leq r} a_{i,j} \lambda_{m,i} \lambda_{m,j} + \sum_{1 \leq j < r} b_j x_{m,j} + \sum_{1 \leq i \leq r, 1 \leq j \leq s} c_{i,j} \lambda_{m,i} y_j + \sum_{1 \leq i < j \leq s} d_{i,j} y_i y_j + \sum_{1 \leq j \leq s} e_j z_j$$

for some constants $\alpha, a_{i,j}, b_j, c_{i,j}, d_{i,j}, e_j \in k$ which are not all zero. Since $\operatorname{Ker} \psi^*$ is stable under the Steenrod operations and $\mathscr{P}^0(x_{m,r}) = 0$, we conclude that $\mathscr{P}^0(u) = 0$, which forces $u$ to be of the form

$$(10.2) \qquad u = \alpha \zeta_m^2 + \lambda_{m,r} \Big( \sum_{1 \leq i < r} a_i \lambda_{m,i} \Big) + \lambda_{m,r} \Big( \sum_{1 \leq j \leq s} c_j y_j \Big).$$

Proposition 4.1 together with Cartan formula imply (by induction) that

$$(10.3) \qquad \mathscr{P}^i(\zeta_m^{2i}) = \zeta_m^{2pi}$$

and all other Steenrod operations vanish on $\zeta_m^{2i}$.

Since $\beta \mathscr{P}^0(\lambda_{m,r}) = \zeta_m^2$, applying $\beta \mathscr{P}^0$ to (10.2), we get

$$(10.4) \qquad \zeta_m^2 \Big( \sum_{1 \leq i < r} a_i^p \lambda_{m,i+1} \Big) + \zeta_m^2 \Big( \sum_{1 \leq j \leq s} c_j^p z_j \Big).$$

Applying $\beta \mathscr{P}^1$, we get

$$(10.5) \qquad \zeta_m^{2p} \Big( \sum_{1 \leq i < r-1} a_i^{p^2} \lambda_{m,i+2} \Big) + \zeta_m^{2p+2} + \zeta_m^{2p} \Big( \sum_{1 \leq j \leq s} c_j^{p^2} z_j \Big).$$

If there is a nonzero coefficient $c_j$, then applying $\mathscr{P}^p, \mathscr{P}^{p^2}, \ldots$ and then taking invariants under the Frobenius map as in the proof of Theorem 4.3, we eventually get that the kernel of the map $\psi^* \colon H^{2,*}(A, k) \to H^{2,*}(G, k)$ contains an element $\zeta_m^{2p^i} \beta \mathscr{P}^0(y)$ with

$y \in H^1((\mathbb{Z}/p)^{\times s}, \mathbb{F}_p) \subset H^{1,0}(\overline{G}, k) \cong H^{1,0}(A, k)$. This means that $f^*$ vanishes on $\zeta^{2p^i}\beta\mathscr{P}^0(y)$, contradicting assumption (4). Hence, we can assume that all coefficients $c_j$ are zero.

Suppose there is a coefficient $a_i \neq 0$. Then (10.5) has the form

$$(10.6) \qquad \zeta_m^{2p}\Big(\sum_{1 \leq i < r-1} a_i^{p^2}\lambda_{m,i+2}\Big) + \zeta_m^{2p+2}.$$

Applying $\mathscr{P}^p, \mathscr{P}^{p^2}, \ldots$ and stopping right before everything annihilates, we conclude that $\zeta_m^{2p^i+2} \in \operatorname{Ker}\psi^*$, once again contradicting assumption (4).

Hence, all coefficients, except for possibly $\alpha$, are zero. This proves the claim. $\qquad\square$

Since $G/G_{\mathsf{ev}} \cong A/A_{\mathsf{ev}} \cong \mathbb{G}_a^-$, the extension (10.1) restricts to an extension on the even subgroup schemes.

$$(10.7)$$



This gives rise to a commutative diagram of the corresponding 5-term sequences:



By Claim 10.3.2, $d_2(\lambda) = x_{m,r} + \alpha\zeta^2$. Since $\zeta$ goes to 0 under the restriction map $H^{*,*}(A, k) \to H^*(A_{\mathsf{ev}}, k)$, we get that $d_2^{\mathsf{ev}}(\lambda) = x_{m,r} \in H^2(A_{\mathsf{ev}}, k) = H^2(E_{m,r}, k)$.

Consider the standard surjection map: $E_{m+1,r} \times (\mathbb{Z}/p)^{\times s} \to E_{m,r} \times (\mathbb{Z}/p)^{\times s}$. By Theorem 9.9, $d_2^{\mathsf{ev}}(\lambda) = x_{m,r}$ vanishes when inflated to $H^2(E_{m+1,r}, k)$. Proposition 10.2 now implies that $G_{\mathsf{ev}}/N$ is abelian. This contradicts the choice of $N$, and completes the proof that $G = A$ in this case.

It remains to consider the case $m = 1$, that is, when $A \cong E_{1,r}^- \times (\mathbb{Z}/p)^{\times s} = \overline{G}$. In this case $d_2(\lambda) = \zeta^2 + \gamma x_r$, and, hence, $d_2^{\mathsf{ev}}(\lambda) = \gamma x_r$. Considering the surjective map $E_{2,r} \times (\mathbb{Z}/p)^{\times s} \to \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^{\times s}$, we conclude by Proposition 10.2 that $G_{\mathsf{ev}}/N$ is abelian, getting a contradiction again. Hence, $G = A$ in the case $m = 1$.

*Case II:* $A \cong E_{m,r+1,\mu}^- \times (\mathbb{Z}/p)^{\times s}$. The proof is very similar, replacing $x_{m,r}$ with $x_{m,r+1,\mu}$ from Theorem 9.10. The corresponding abelian cover which plays the role of $A'$ in Proposition 10.2 in this case is the canonical map $E_{m,r+1} \to E_{m,r+1,\mu}$. $\qquad\square$

## 11. The main detection theorem

The proof of the main detection Theorem 1.2 effectively splits into two parts. The first part covers the case when $G$ satisfies Hypothesis 7.1. The techniques needed to deal with this case are mostly adaptations of what was done for finite group schemes (without the grading) and are summarized in Part I of the paper. The only, but significant, exception is Theorem 4.4 which requires extensive new calculations for cohomology of supergroup schemes done in [14]. In the ungraded case the only group schemes which fail Hypotheses 7.1 are the elementary ones, that is, finite groups schemes isomorphic to $\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^{\times s}$, which form

the detection family. Hence, the inductive detection Theorem 7.2 gives the full detection theorem in the ungraded case.

In the super case however we have to deal with case (ii) of Theorem 4.3 when the kernel of the map on cohomology induced by $f\colon G \to \overline{G}$ has an element of the form $\zeta^2 - \gamma x_r$. The new technology developed in Part II culminating in the cohomological characterization of the elementary supergroup schemes in Theorem 10.3 is what we need to deal with this case.

Theorem 1.2 is an immediate consequence of the following theorem. We employ terminology of a *detection family* introduced in the beginning of Section 6.

**Theorem 11.1.** *Suppose that $G$ is a finite unipotent supergroup scheme which is not isomorphic to a quotient of some $E_{m,n}^- \times (\mathbb{Z}/p)^{\times s}$. Then*

> (i) *nilpotence of elements in cohomology of modules and*
> (ii) *projectivity of $kG$-modules*

*are detected on proper sub-supergroup schemes after field extension.*

*Proof.* Let $G = G^0 \rtimes \pi$ with $G^0$ connected and $\pi$ finite. Since $G$ is unipotent, so is $G^0$, and $\pi$ is a finite $p$-group. If $\pi$ is not elementary abelian, then by Theorem 4.2, $G$ satisfies case (c) of Hypothesis 7.1, and we are done. So we now assume that $\pi = (\mathbb{Z}/p)^{\times s}$ is elementary abelian. By Lemma 3.1,

$$H^{1,0}(G,k) \cong \operatorname{Hom}_{\mathsf{sGr}/k}(G^0, \mathbb{G}_a)^\pi \times \operatorname{Hom}(\pi, \mathbb{G}_a)$$

$$H^{1,1}(G,k) \cong \operatorname{Hom}_{\mathsf{sGr}/k}(G^0, \mathbb{G}_a^-)^\pi.$$

We examine the dimensions

$$\delta = \dim_k \operatorname{Hom}_{\mathsf{sGr}/k}(G^0, \mathbb{G}_{a(1)})^\pi$$

$$\varepsilon = \dim_k \operatorname{Hom}_{\mathsf{sGr}/k}(G^0, \mathbb{G}_a^-)^\pi.$$

Since $\pi$ is unipotent, if $\delta = 0$ then $\operatorname{Hom}_{\mathsf{sGr}/k}(G^0, \mathbb{G}_{a(1)}) = 0$, and if $\varepsilon = 0$ then $\operatorname{Hom}_{\mathsf{sGr}/k}(G^0, \mathbb{G}_a^-) = 0$. Thus, $\delta = \epsilon = 0$, then $G^0$ is trivial by Lemma 3.2, hence, $G \cong (\mathbb{Z}/p)^{\times s}$, and we are done. We may therefore assume that one of them is nonzero. If either $\delta$ or $\varepsilon$ is greater than one then we are in case (a) or (b) of Hypothesis 7.1, and we are done by Theorem 7.2. So each is either zero or one, and they are not both zero.

The action of the Frobenius map $F\colon \mathbb{G}_a \to \mathbb{G}_a$ induces a map

$$F\colon \operatorname{Hom}_{\mathsf{sGr}/k}(G^0, \mathbb{G}_a) \to \operatorname{Hom}_{\mathsf{sGr}/k}(G^0, \mathbb{G}_a)$$

which commutes with the action of $\pi$. A $\pi$-invariant map $G^0 \to \mathbb{G}_a$ lands in $\mathbb{G}_{a(1)} \le \mathbb{G}_a$ if and only if it is in the kernel of $F$. So there exists $r \ge 0$ and a surjective map

$$\xi \in \operatorname{Hom}_{\mathsf{sGr}/k}(G^0, \mathbb{G}_{a(r)})^\pi$$

such that $\xi, F(\xi), \ldots, F^{r-1}(\xi)$ is a $k$-basis for $\operatorname{Hom}_{\mathsf{Gr}/k}(G^0, \mathbb{G}_a)^\pi$. The map $\xi$ extends to a surjective map

$$f\colon G \to \overline{G} \cong \mathbb{G}_{a(r)} \times (\mathbb{G}_a^-)^\varepsilon \times (\mathbb{Z}/p)^{\times s}$$

and $f^*\colon H^{1,*}(\overline{G}, k) \to H^{1,*}(G, k)$. This construction accounts both for the case $\delta = 0$ (with $r = 0$ so that $\overline{G} = \mathbb{G}_a^- \times (\mathbb{Z}/p)^{\times s}$) and $\epsilon = 0$ (with $\overline{G} = \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^{\times s}$, $r \ge 1$).

If $f$ is an isomorphism then $G \cong \overline{G}$ is isomorphic to a quotient of $E_{1,r}^- \times (\mathbb{Z}/p)^{\times s}$ contradicting the assumption of the theorem. Otherwise, by Lemma 3.5,

$$f^*\colon H^{2,*}(\overline{G}, k) \to H^{2,*}(G, k)$$

is not injective. If the kernel contains an element of degree $(2, 1)$, then by Theorem 4.4 we are in case (c) of Hypothesis 7.1, so we are done by Therefore, we may assume that the kernel contains an element of degree $(2, 0)$ and we have two cases according to Theorem 4.3. In the first case, it contains an element of the form

$$x_r^n \beta \mathscr{P}^0(v_1) \ldots \beta \mathscr{P}^0(v_m),$$

which again puts us in case (c) of Hypothesis 7.1, and we again apply Theorem 7.2. In the second case, the kernel is generates by $\zeta^2 + \gamma x_r$. If $\gamma = 0$, then we can apply Theorem 7.2 once again, since Hypothesis 7.1 is satisfied by the image of $\zeta^2$.

The upshot of this is that we may assume that we are in case (ii) of Theorem 4.3 with $\gamma \neq 0$ and that $f^*$ induces an isomorphism on $H^{2,*}$. Hence, $G$ satisfies the hypotheses (1), (2) and (3) of Theorem 10.3. If it fails hypothesis (4) of Theorem 10.3, then we are in case (c) of Hypothesis 7.1 one last time. Otherwise, $G$ is isomorphic to a quotient of $E_{m,r}^- \times (\mathbb{Z}/p)^{\times s}$ for some $m \geq 2$ by Theorem 10.3. $\qquad \square$

There is another notion of nilpotency for elements of $H^{*,*}(G, \Lambda)$ where $\Lambda$ is a unital $G$-algebra. Namely, $\xi \in H^{i,*}(G, \Lambda)$ is nilpotent if for some $n > 0$, the image of $\xi^{\otimes n} \in H^{in,*}(G, \Lambda^{\otimes n})$ in $H^{in,*}(G, \Lambda)$ is zero. The following analogue of Theorem 11.1 for this notion of nilpotents has both a weaker hypothesis and a weaker conclusion than (11.1).

**Theorem 11.2.** *Let $G$ be a finite unipotent supergroup scheme over a field $k$, and $\Lambda$ be unital $G$-algebra. Then an element $x \in H^{i,*}(G, \Lambda)$ is nilpotent, that is $x^n \in H^{in,*}(G, \Lambda)$ is zero for some $n > 0$, if and only if for every extension field $K$ of $k$ and every elementary sub-supergroup scheme $E$ of $G_K$, the restriction of $x_K \in H^{*,*}(G_K, \Lambda_K)$ to $H^{*,*}(E, \Lambda_K)$ is nilpotent, that is some power of $x_K$ vanishes in $H^{*,*}(E, \Lambda_K)$.*

*Proof.* First, we claim that the analogue of Theorem 7.2 holds for $H^{*,*}(G, \Lambda)$ with this notion of nilpotency. Indeed, If we take $M = \Lambda$ in Proposition 5.3 then the conclusion clearly holds for $\xi^2 \in H^{*,*}(G, \Lambda)$. Hence, if $G$ satisfies Hypothesis 7.1(c), the proof of Theorem 7.2 carries over to this case.

If we assume that Hypotheses 7.1 (a) or (b) hold, then the proof is identical to that of Case II(b) in [3, Theorem 6.1] (see also [6, Theorem 2.5]) so we will not reproduce it here.

With these observations, the proof of the analogue of Theorem 11.1 is again identical to the one we give above. $\qquad \square$

In [10], we show that projectivity for modules of finite group schemes is detected on the family of elementary subgroup schemes after *coextension* of scalars. In the following theorem we state that this also holds for finite unipotent supergroup schemes.

**Theorem 11.3.** *Let $G$ be a finite unipotent supergroup scheme over a field $k$ of positive characteristic $p > 2$, and $M$ be a $kG$-module. Then the following hold.*

(i) *An element $x \in H^{*,*}(G, M)$ is nilpotent if and only if for every extension field $K$ of $k$ and every elementary sub-supergroup scheme $E$ of $G_K$, the restriction of $x_K \in H^{*,*}(G_K, M^K)$ to $H^{*,*}(E, M^K)$ is nilpotent.*

(ii) *A $kG$-module $M$ is projective if and only if for every extension field $K$ of $k$ and every elementary sub-supergroup scheme $E$ of $G_K$, the restriction of $M^K$ to $E$ is projective.*

*Proof.* The proof of Theorem 11.1 carries over to this case almost without change. The only difference occurs when $G$ satisfies (a) or (b) of Hypothesis 7.1. Then we still proceed exactly

39

as in [3, Theorem 8.1] but appeal to [10, Lemma 4.1] for the main ingredient which is the appropriate version of the Kronecker quiver lemma for $M^K$. □

## 12. The Steenrod algebra

Algebraic topology often deals with $\mathbb{Z}$-graded commutative or $\mathbb{Z}$-graded cocommutative Hopf algebras, where the $\mathbb{Z}$-grading introduces the usual sign conventions.

Let $k$ be a $\mathbb{Z}$-*graded field* of odd characteristic $p$. This means that either $k$ is a field in the usual sense, concentrated in degree zero, or it is a ring of Laurent polynomials $k = k_0[u, u^{-1}]$, where $u$ has nonzero degree, which is even if $k_0$ does not have characteristic two. Over a $\mathbb{Z}$-graded field, every graded module is free. This means that it is isomorphic to a direct sum of shifts of $k$.

An *affine $\mathbb{Z}$-graded group scheme* over $k$ is a covariant functor from $\mathbb{Z}$-graded commutative $k$-algebras (again, the convention is that $yx = (-1)^{|x||y|}xy$) to groups, whose underlying functor to sets is representable. If $G$ is an affine $\mathbb{Z}$-graded group scheme over $k$ then its coordinate ring $k[G]$ is the representing object. It is a $\mathbb{Z}$-graded commutative Hopf algebra. This gives a contravariant equivalence of categories between affine $\mathbb{Z}$-graded group schemes and $\mathbb{Z}$-graded commutative Hopf algebra.

An affine $\mathbb{Z}$-graded group scheme $G$ has *finite type* if each graded piece is finite dimensional. In this case, the graded dual $kG^i = \mathrm{Hom}_k(k[G]^{-i}, k)$ is a $\mathbb{Z}$-graded cocommutative Hopf algebra of finite type. This gives a covariant equivalence of categories between $\mathbb{Z}$-graded group schemes of finite type and $\mathbb{Z}$-graded cocommutative Hopf algebras of finite type.

We are interested in particular in the *finite* $\mathbb{Z}$-graded group schemes; these are the ones for which not only is each graded piece finite dimensional, but the total rank as a $k$-module is finite. In other words, if $k$ is a field in degree zero, this means that there are only finitely many degrees in which $k[G]$ is nonzero, whereas if $k$ is a ring of Laurent polynomials, it is equivalent to finite type.

**Theorem 12.1.** *Let $G$ be a finite $\mathbb{Z}$-graded unipotent group scheme over a $\mathbb{Z}$-graded field $k$, and $M$ be a $kG$-module. Then the following hold.*

(i) *An element $\xi$ of $H^{*,*}(G, M)$ is nilpotent if and only if for every extension $\mathbb{Z}$-graded field $K$ of $k$, and every elementary subgroup scheme $E$ of $G_K$, the restriction of $\xi_K \in H^{*,*}(G_K, M_K)$ to $H^{*,*}(E, M_K)$ is nilpotent.*

(ii) *A $kG$-module $M$ is projective if and only if for every extension $\mathbb{Z}$-graded field $K$ of $k$, and every elementary subgroup scheme $E$ of $G_K$, the restriction of $M_K$ to $E$ is projective.*

*Proof.* If $K = K_0[u, u^{-1}]$ with $|u| = 2$, there is a one to one correspondence between finite $\mathbb{Z}$-graded group schemes $G$ over $K$ and finite supergroup schemes $\overline{G}$ over $K_0$. This is given by identifying the even graded pieces using powers of $u$, and similarly with the odd graded pieces. Similarly if $M$ is a $KG$-module then $\overline{M}$ is a $K_0\overline{G}$-module. Furthermore, the space of $KG$-module homomorphisms $\mathrm{Hom}^i_{KG}(M, N)$ of degree $i$ is isomorphic to $\mathrm{Hom}^{\overline{i}}_{K_0\overline{G}}(\overline{M}, \overline{N})$ where $\overline{i}$ is the reduction of $i$ modulo two. So if we use powers of $u$ to identify an even shift of $M$ with $M$, thus giving $\mathsf{Mod}(KG)$ a mod two internal shift, there is an equivalence of categories between $\mathsf{Mod}(KG)$ and $\mathsf{Mod}(K_0\overline{G})$ preserving the internal shift modulo two. Projectivity is therefore preserved, and $\mathsf{StMod}(KG)$ is equivalent to $\mathsf{StMod}(K_0\overline{G})$. Similarly,

$H^{*,*}(\overline{G}, K_0)$ is obtained by reducing the internal grading modulo two on $H^{*,*}(G, K)$, again using powers of $u$.

Since every extension $\mathbb{Z}$-graded field of $k$ can be extended to one of the type $K = K_0[u, u^{-1}]$ above, the theorem follows from Theorem 1.2. $\qquad\square$

Recall from Milnor [36], Steenrod and Epstein [41] that for $p$ odd, the graded dual $\mathcal{A}^*$ of the Steenrod algebra $\mathcal{A}$ is a tensor product

$$k[\xi_1, \xi_2, \dots] \otimes \Lambda(\tau_0, \tau_1, \dots)$$

of a polynomial ring in generators $\xi_n$ of degree $2p^n - 2$ and an exterior algebra in generators $\tau_n$ of degree $2p^n - 1$. We also set $\xi_0 = 1$. With this notation, the comultiplication is given by

$$\Delta(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i, \qquad \Delta(\tau_n) = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \tau_i.$$

If $A$ is a finite dimensional Hopf subalgebra of $\mathcal{A}$ then the graded dual $A^*$ is a finite dimensional quotient of $\mathcal{A}^*$. Let $G$ be the finite $\mathbb{Z}$-graded group scheme corresponding to $A$, so that $\mathbb{F}_p G \cong A$ and $\mathbb{F}_p[G] \cong A^*$. Then $\mathbb{F}_p[G_{(1)}]$ is a quotient of $\mathcal{A}^*$ by a Hopf ideal containing $\xi_1^p, \xi_2^p, \dots$. Letting $\bar{\xi}_n$ and $\bar{\tau}_n$ be the images of $\xi_n$ and $\tau_n$ in this quotient, for $n \geq 1$ we have

$$\Delta(\bar{\xi}_n) = \bar{\xi}_n \otimes 1 + 1 \otimes \bar{\xi}_n, \qquad \Delta(\bar{\tau}_n) = \bar{\tau}_n \otimes 1 + 1 \otimes \bar{\tau}_n + \bar{\xi}_n \otimes \bar{\tau}_0$$

while $\Delta(\bar{\tau}_0) = \bar{\tau}_0 \otimes 1 + 1 \otimes \bar{\tau}_0$. In other words, $\bar{\xi}_n$ ($n \geq 1$) and $\bar{\tau}_0$ are primitive, and $\bar{\tau}_n$ ($n \geq 1$) are primitive modulo $\bar{\tau}_0$.

If we isolate a single $n$, and dualise these relations for $\bar{\xi}_n$, $\bar{\tau}_n$ and $\bar{\tau}_0$ we get the restricted universal enveloping algebra of a three dimensional restricted Lie superalgebra consisting of the matrices

$$\left( \begin{array}{cc|c} 0 & * & * \\ 0 & 0 & * \\ \hline 0 & 0 & 0 \end{array} \right)$$

in $GL(2|1)$. The dual elements $\bar{\xi}_n^*$ and $\bar{\tau}_n^*$ to $\bar{\xi}_n$ and $\bar{\tau}_n$ are in the top row, and the dual element $\bar{\tau}_0^*$ to $\bar{\tau}_0$ is in the second row. The only non-trivial commutator relation is $[\bar{\xi}_n^*, \bar{\tau}_0^*] = \bar{\tau}_n^*$.

Dualising, we get a homomorphism $G_{(1)} \to \mathbb{G}_a^-$, and the kernel is isomorphic to a subgroup scheme of $(\mathbb{G}_{a(1)})^{\times s} \times (\mathbb{G}_a^-)^r$. Every subgroup scheme again has this form, so we have proved the following lemma.

**Lemma 12.2.** *Let $A$ be a finite dimensional Hopf subalgebra of the Steenrod algebra, and let $G$ be the corresponding $\mathbb{Z}$-graded group scheme. Then there is a (possibly trivial) homomorphism $G_{(1)} \to \mathbb{G}_a^-$ whose kernel is isomorphic to $(\mathbb{G}_{a(1)})^{\times s} \times (\mathbb{G}_a^-)^r$ for some $r, s \geq 0$. The $(\mathbb{G}_a^-)^r$ is normal, and the quotient is commutative. In particular, there is no sub-supergroup scheme isomorphic to $W_{m,1}^-$ for $m \geq 1$.* $\qquad\square$

Conceptually, what we have done amounts to showing that the first Frobenius kernel of the Steenrod algebra is an extension of $\mathbb{G}_a^-$ by an infinite product of copies of $\mathbb{G}_{a(1)} \times \mathbb{G}_a^-$, with gradings tending to infinity, in such a way that over each factor the extension is the one described by the above subgroup of $GL(2|1)$.

**Proposition 12.3.** *Let $A$ be a finite dimensional sub Hopf algebra of the Steenrod algebra $\mathcal{A}$ over $\mathbb{F}_p$, and let $G$ be the corresponding finite unipotent connected $\mathbb{Z}$-graded group scheme. If $E$ is an elementary $\mathbb{Z}$-graded subgroup scheme of $G$ then $E \cong \mathbb{G}_{a(n)} \times \mathbb{G}_a^-$.*

*Proof.* By Theorem 8.13, we have that $E \cong E_{m,n}^-$ or $E \cong E_{m,n,\mu}^-$. We need to show that $m = 1$. But if $m \geq 2$, the statement follows from the observation that $(E_{m,n}^-)_{(1)}$ and $(E_{m,n,\mu}^-)_{(1)}$ both contain $W_{m-1,1}^-$ as a subgroup scheme. But $(\mathbb{G}_{a(1)})^{\times s} \times (\mathbb{G}_a^-)^{\times r}$ does not, and therefore by Lemma 12.2 neither does $G_{(1)}$. $\square$

The detection theorem for the finite dimensional subalgebras of the Steenrod algebra now follows from Theorem 12.1 and Proposition 12.3.

**Theorem 12.4.** *Let $A$ be a finite dimensional sub-Hopf algebra of the Steenrod algebra $\mathcal{A}$ over $\mathbb{F}_p$. Then $A$ is the group algebra of a $\mathbb{Z}$-graded finite group scheme. The following hold:*

(1) *For an $A$-module $M$, an element $\xi$ in $H^{*,*}(A, M)$ is nilpotent if and only if for every extension $\mathbb{Z}$-graded field $K$ of $k$, the restriction of $\xi_K \in H^{*,*}(A_K, M_K)$ to every subgroup scheme of $A_K$ isomorphic to $\mathbb{G}_{a(r)}$, $\mathbb{G}_a^-$, or $\mathbb{G}_{a(r)} \times \mathbb{G}_a^-$ is nilpotent.*

(2) *An $A$-module $M$ is projective if and only if for every extension $\mathbb{Z}$-graded field $K$ of $k$, the restriction of $M_K$ to every subgroup scheme of $A_K$ isomorphic to $\mathbb{G}_{a(r)}$, $\mathbb{G}_a^-$, or $\mathbb{G}_{a(r)} \times \mathbb{G}_a^-$ is projective.* $\square$

Nakano and Palmieri in [37] prove a detection theorem specifically for finite dimensional subalgebras of the Steenrod algebra. Their result differs from ours in two aspects: their detection family of "quasi–elementary" subalgebras is substantially bigger than ours, but on the other hand their result does not require extension of scalars. The results can be deduced from each other using the methods developed in [37] and in this paper. The key feature of our main result as well as of its application to the Steenrod algebra, is that our detecting family is minimal. Our work also demonstrates that the relatively simple detection family which works for finite dimensional sub Hopf algebras of the Steenrod algebra is an anomaly rather than the norm for general finite dimensional graded cocommutative Hopf algebras.

## Appendix A. Witt vectors and Dieudonné modules

Recall that finite commutative connected unipotent group schemes form an abelian category $\mathfrak{A}$ which is equivalent to an appropriate category of Dieudonné modules. This is described for example in Fontaine [26], but we give an outline here. What will interest us is the Dieudonné modules killed by $p$, which were classified by Koch [31].

We begin with a brief recollection concerning the Witt vectors. Define a polynomial $w_n$ in variables $Z_0, \ldots, Z_n$ with integer coefficients by

$$w_n(Z_0, \ldots, Z_n) = p^n Z_n + p^{n-1} Z_{n-1}^p + \cdots + Z_0^{p^n}.$$

Then the polynomials $S_i$ and $P_i$ in variables $X_0, \ldots, X_n, Y_0, \ldots, Y_n$, again with integer coefficients, are defined by

$$w_n(S_0, \ldots, S_n) = w_n(X_0, \ldots, X_n) + w_n(Y_0, \ldots, Y_n),$$
$$w_n(P_0, \ldots, P_n) = w_n(X_0, \ldots, X_n)w_n(Y_0, \ldots, Y_n).$$

So for example $S_0 = X_0 + Y_0$, $P_0 = X_0 Y_0$,

$$S_1 = X_1 + Y_1 + \frac{(X_0 + Y_0)^p - X_0^p - Y_0^p}{p}, \qquad P_1 = pX_1 Y_1 + X_0^p Y_1 + X_1 Y_0^p,$$

and so on.

Witt vectors $W(k)$ over $k$ are vectors $(a_0, a_1, \dots)$ with $a_i \in k$, where $S_i$ and $P_i$ give the coordinates of the sum and product:

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = (S_0(a_0, b_0), S_1(a_0, a_1, b_0, b_1), \dots)$$
$$(a_0, a_1, \dots)(b_0, b_1, \dots) = (P_0(a_0, b_0), P_1(a_0, a_1, b_0, b_1), \dots).$$

Thus for example if $k = \mathbb{F}_p$ then $W(k)$ is the ring of $p$-adic integers $\mathbb{Z}_p$. More generally, $W(k)$ is a local ring of mixed characteristic $p$. The Frobenius endomorphism of $k$ lifts to a ring endomorphism of $W(k)$ denoted $\boldsymbol{\sigma}$. It is defined by $(a_0, a_1 \dots)^{\boldsymbol{\sigma}} = (a_0^p, a_1^p, \dots)$.

More generally, if $A$ is a commutative $k$-algebra then $W(A)$ is the ring of Witt vectors over $A$, defined using the same formulae. This defines a functor from commutative $k$-algebras to rings. The additive part of this functor defines an affine group scheme over $k$ denoted $W$, the *additive Witt vectors*. If we stop at length $m$ vectors, we obtain $W_m$, and we write $W_{m,n}$ for the $n$th Frobenius kernel of $W_m$.

There are two endomorphisms $V$ and $F$ of $W$ of interest to us. These are the Verschiebung $V$ defined by

$$V(a_0, a_1, \dots) = (0, a_0, a_1, \dots),$$

and the Frobenius $F$ given by

$$F(a_0, a_1, \dots) = (a_0^p, a_1^p, \dots).$$

These commute, and their product corresponds to multiplication by $p$ on Witt vectors. Multiplication by a Witt vector $x \in W(k)$ also gives an endomorphism of $W$ which we shall denote $x$ by abuse of notation. These are related to $V$ and $F$ by the relations $Vx^{\boldsymbol{\sigma}} = xV$ and $Fx = x^{\boldsymbol{\sigma}} F$.

We write $W_m$ for the group scheme of Witt vectors of length $m$, corresponding to the quotient $W(k)/(p^m)$ of $W(k)$. This is a group scheme with a filtration whose quotients are $m$ copies of the additive group $\mathbb{G}_a$. We write $W_{m,n}$ for the $n$th Frobenius kernel of $W_m$. This is a finite group scheme with a filtration of length $mn$ whose quotients are copies of $\mathbb{G}_{a(1)}$.

The *Dieudonné ring* $D_k$ is generated over $W(k)$ by two commuting variables $V$ and $F$ satisfying the following relations:

$$FV = VF = p, \qquad Vx^{\boldsymbol{\sigma}} = xV, \qquad Fx = x^{\boldsymbol{\sigma}} F$$

for $x \in W(k)$. Then $W$ is a module over $D_k$, as are its quotients $W_m$ and their finite subgroup schemes $W_{m,n}$.

Recall that there is a duality on $\mathfrak{A}$ called Cartier duality, which corresponds to taking the $k$-linear dual of the corresponding Hopf algebras. We denote the Cartier dual of $G$ by $G^\sharp$.

Now consider the subcategory $\mathfrak{A}_{m,n}$ of $\mathfrak{A}$ consisting of the those group schemes $G$ in $\mathfrak{A}$ such that $G$ has height at most $n$ and the Cartier dual $G^\sharp$ has height at most $m$. Then there is a covariant equivalence of categories between $\mathfrak{A}_{m,n}$ and the category $\mathsf{mod}(D_k/(V^m, F^n))$ of finite length modules over the quotient ring $D_k/(V^m, F^n)$. This equivalence is given by the functor

$$\mathrm{Hom}_{\mathfrak{A}}(W_{m,n}, -) \colon \mathfrak{A}_{m,n} \to \mathsf{mod}(D_k/(V^m, F^n)).$$

Write $\hat{D}_k$ for the corresponding completion $\varprojlim D_k/(V^m, F^n)$ Then every $\hat{D}_k$-module of finite length is a module for some quotient of the form $D_k/(V^m, F^n)$, and these equivalences combine to give an equivalence between $\mathfrak{A}$ and the category $\mathsf{fl}(\hat{D}_k)$ of $\hat{D}_k$-modules of finite length. Let us write

$$\psi\colon \mathsf{fl}(\hat{D}_k) \to \mathfrak{A}$$

for this equivalence. Thus for example $\psi(D_k/(V^m, F^n)) \cong W_{m,n}$ and $\psi(D_k/(V^m, F^n, p)) \cong W_{m,n}/W_{m-1,n-1} \cong E_{m,n}$, where the last notation is introduced in Defintion 8.6.

Let $G = \psi(M)$ be a finite unipotent abelian group scheme, so that $M$ is a finite length $D_k/(V^m, F^n)$-module for some $m, n \geq 1$. If we are only interested in the algebras structure of $G$, this means that we can ignore the action of $F$ on $M$ and just look at finite length modules for $D_k/(V^m, F) = W(k)[V]$ with $xV = Vx^\sigma$ ($x \in W(k)$). Such modules are always direct sums of cyclic submodules, and the cyclic modules are just truncations at smaller powers of $V$. Translating through the equivalence $\psi$, we have the following.

**Lemma A.1.** *Let $G$ be a finite unipotent abelian group scheme. Then $kG$ is a isomorphic to a tensor product of algebras of the form $kW_{m,1} \simeq k[s]/s^{p^m}$.*

**Lemma A.2.** *Let $G$ be a finite unipotent abelian group scheme. If $\dim_k \operatorname{Hom}_{\mathsf{Gr}/k}(G, \mathbb{G}_{a(1)}) = 1$ and $G$ does not have $W_{2,2}$ as a quotient, then $G$ is isomorphic to a quotient of the group scheme $E_{m,n}$.*

*Proof.* The condition $\dim_k \operatorname{Hom}_{\mathsf{Gr}/k}(G, \mathbb{G}_{a(1)}) = 1$ implies that the corresponding Dieudonné module is cyclic, $G_{\mathsf{ev}} \cong \psi(D_k/I)$ for some ideal $I$ containing $V^m$ and $F^n$ for some $m, n$. Not having $W_{2,2}$ as a quotient implies that $p = FV$ kills $D_k/I$, and, hence, $G$ is isomorphic to a quotient $D_k/(V^m, F^n, p)$. But the latter is precisely $E_{m,n}$. $\square$

The last thing we need is the classification of the quotients of the group scheme $E_{m,n}$. In terms of Dieudonné modules, we have

$$E_{m,n} = \psi(D_k/(V^m, F^n, p)).$$

The isomorphism classes of quotients of $D_k/(V^m, F^n, p)$ were classified by Koch [31]. The main results of that paper may be stated as follows.

**Theorem A.3.** *Every nonzero finite quotient of $\hat{D}_k/(p)$ as a left $\hat{D}_k$-module is isomorphic to either $M_{m,n} = D_k/(V^m, F^n, p)$ (of length $m + n - 1$) or $M_{m,n,\mu} = D_k/(F^n - \mu V^m, p)$ (of length $m + n$) for some $m, n \geq 1$ and $0 \neq \mu \in k$. The only isomorphisms among these modules are given by $M_{m,n,\mu} \cong M_{m,n,\mu'}$ if and only if $\mu/\mu' = a^{p^{m+n}-1}$ for some $a \in k$.*

*Outline of proof.* Let $M$ be a nonzero finite quotient of $\hat{D}_k/(p)$, let $m$ be the height of $M^\sharp$ and $n$ be the height of $M$. Then $M$ is a finite quotient of $D_k/(V^m, F^n, p)$. So either $M$ is isomorphic to $D_k/(V^m, F^n, p)$ or the kernel is at least one dimensional. If the kernel has length one, then it is in the socle, which has length two, and is the image of $V^{m-1}$ and $F^{n-1}$. By minimality of $m$ and $n$, the kernel is then $(F^{n-1} - \mu V^{m-1})$ for some $0 \neq \mu \in k$. If $M$ is equal to this, we have $M \cong M_{m-1,n-1,\mu}$. Otherwise $M$ is a proper quotient of $M_{m-1,n-1,\mu}$. But the socle of $M_{m-1,n-1,\mu}$ is one dimensional, spanned by the image of $V^{m-1}$, so in this case $M$ is a quotient of $M_{m-1,n-1}$, which implies that $m$ and $n$ are not minimal. This contradiction proves that these are the only isomorphism types.

The dimensions of $M/F^i M$ and $M/V^i M$ distinguish all isomorphism classes, with the possible exception of isomorphisms between $M_{m,n,\mu}$ and $M_{m,n,\mu'}$. Such an isomorphism is determined modulo radical endomorphisms by a scalar $a \in k^\times \subseteq W(k)^\times$. The equation $(F^n - \mu V^m)a = b(F^n - \mu' V^m)$ implies that $b = a^{\sigma^n}$ and $\mu a = b^{\sigma^m} \mu'$. Thus

$$\mu/\mu' = a^{\sigma^{m+n}} a^{-1} = a^{p^{m+n}-1}. \qquad \square$$

*Remark* A.4. Note that if $k = \mathbb{F}_p$ then this condition on $\mu$ and $\mu'$ is only satisfied if $\mu = \mu'$, so there are $p - 1$ isomorphism classes of $M_{m,n,\mu}$. But if $k$ is algebraically closed then the isomorphism type of $M_{m,n,\mu}$ is independent of $\mu$.

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