# POINT GROUPS

LECTURE 8, EXERCISE SET 1

Recall that we denote by $M$ the group of all rigid motions of the plane. An (orthogonal) subgroup $\mathbb{O} < M$ is the subgroup of all motions which fix the origin. A subgroup $T < M$ is the subgroup of all translations of the plane.

**Definition.** A subgroup of $M$ is called *discrete* if it does not contain arbitrarily small rotations or translations.

**Exercise 1.** Show that a discrete subgroup of $M$ consisting of rotations around the origin is cyclic and is generated by some rotation $\rho_\theta$.

**Exercise 2.** Show that a discrete subgroup of $\mathbb{O}$ is a finite group.

**Definition.** Let $G$ be a discrete group of rigid motions of the plane. The *translation subgroup* of $G$ is the subgroup generated by all translations $t_{\vec{a}}$ in $G$.

**Example.** Let $G$ be a group of symmetries of an infinite checkboard table (with the basis square $1 \times 1$). The translation subgroup of $G$ is the lattice on the basis vectors $(1, 0)$ and $(0, 1)$.

There are three possibilities: $L$ is trivial (then $G$ is a finite group of rigid motions), $L$ is generated by just one translation (this leads to frieze patterns), and $L$ is generated by two indepenent translations.

**Definition.** A discrete group of rigid motions of the plane is called a 2-dimensional **crystallographic group** if the subgroup $L$ of $G$ is a lattice, i.e., $L$ is generated by two linearly independent vectors $\vec{a}$, $\vec{b}$.

There is one-to-one correspondence:

$$\text{Wallpaper patterns} \longleftrightarrow \text{Crystallographic groups}$$

In the next exercise we prove the theorem known as *crystallographic restriction.*

**Exercise 3.** Let $H < \mathbb{O}$ be a finite subgroup of the group of symmetries of a lattice $L$. Then

(a) Every rotation in $H$ has order $1, 2, 3, 4$, or $6$.
(b) $H$ is one of the groups $C_n$ or $D_n$ for $n = 1, 2, 3, 4$, or $6$.

Recall that the point group $\overline{G}$ of a crystallographic group $G$ carries the lattice $L$ of $G$ to itself. The point group $\overline{G}$ is also a finite subgroup of $\mathbb{O}$ by Exercise 2. Hence, the last exercise can be reformulated as follows:

**Corollary 4.** *Let $G$ be a 2-dimensional crystallographic group; that is, $G$ is a group of symmetries of a wallpaper pattern. Then the choice for the point group of $G$ (the group which "encodes" all rotations, reflections and glide reflections) is very limited: it is one of the nine (only!!) groups from the list in the Exercise 3b.*

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Hints for Exercise 3a.

(1) Since the group $H$ is finite, some multiple of any angle which defines a rotation in the group $H$ must be $2\pi$.

(2) Choose the shortest vector in $L$ and the smallest angle $\theta$ such that the rotation $\rho_\theta \in H$. Now, if the rotation does not have the specified order, try to construct a vector shorter than the one you chose. How? Well, don't forget that applying rotations from $H$ to a vector from the lattice gives you new vectors from the latttice (since $H$ carries $L$ to itself).