## BACK TO GEOMETRY: MATRICES AS TRANSFORMATIONS OF THE PLANE.

LECTURE 4, EXERCISE SET 1

Exercise 1. Describe geometrically the "action" of each of the matrices below on the plane (for example, the permutation matrix $\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]$ is a reflection through the line $x=y$ )
(1) $\left[\begin{array}{rr}1 & 0 \\ 0 & -1\end{array}\right]$
(2) $\left[\begin{array}{rr}1 & -1 \\ 1 & 1\end{array}\right]$
(3) $\left[\begin{array}{ll}2 & 0 \\ 0 & 2\end{array}\right]$
(4) $\left[\begin{array}{rr}0 & 1 \\ -1 & 0\end{array}\right]$
(5) $\left(\begin{array}{cc}\frac{1}{2} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{2}\end{array}\right)$
(6) $\left(\begin{array}{cc}\frac{1}{\sqrt{3}} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{\sqrt{3}}\end{array}\right)$

This is the end of the first exercise in this set. When you are done, choose one person who may possibly go to the board to write down the answer to one of the exercises. Now raise your hands.

If you have finished ahead of other groups, try to answer the following questions:

- Which of the motions (1)-(6) are orientation preserving?
- Which are orientation reversing?
- Give a necessary and sufficient condition on the matrix $A$ so that the motion of the plane defined by $A$ is orientation-preserving.

Exercise 2. Let $A=\left[\begin{array}{rr}1 & 0 \\ 0 & -1\end{array}\right], B=\left[\begin{array}{rr}0 & -1 \\ 1 & 0\end{array}\right]$ and $e_{2}=\left[\begin{array}{l}0 \\ 1\end{array}\right]$
(1) Compute the vector $v=B e_{2}$
(2) Now compute the vector $A v=A\left(B e_{2}\right)$
(3) Compute the matrix $C=A B$
(4) Now compute $C e_{2}=(A B) e_{2}$
??Did you get the same answer for (2) and (4)??
(5) What are the plane motions defined by the matrices $A, B$ and $C$ ?
(6) What is the composition of the motions corresponding to matrices $A$ and $B$ ?
(7) Can you now answer the question "why we multiply rigid motions from right to left"?

Exercise 3. (a) Write down the matrix which corresponds to a counter-clockwise rotation around the origin by the angle $\phi$.
(b) Let $B$ be the matrix corresponding to a counter-clockwise rotation around the origin by the angle $\phi$, and $A$ be the matrix corresponding to a counter-clockwise rotation around the origin by the angle $\psi$. Then $A B$ corresponds to the rotation by $\phi$ followed by the rotation by $\psi$ which is clearly just a rotation by $\phi+\psi$. Using matrix multiplication, compute $A B$.
(c) The matrix $A B$ you got using matrix multiplication corresponds to the matrix of the rotation by $\phi+\psi$. Hence, you have just proved the laws of $\sin$ and $\cos$ !

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\begin{aligned}
\cos (\phi+\psi) & =\cos \phi \cos \psi-\sin \phi \sin \psi \\
\sin (\phi+\psi) & =\cos \phi \sin \psi+\cos \psi \sin \phi
\end{aligned}
$$

Exercise 4. Write down the matrix which corresponds to a reflection through the line $y=a x$.

Now we try to draw conlusions and make general observations based on computations we have done and evidence they have provided.

Exercise 5. (a) Describe all rigid motions that can be expressed via matrices ( that is, correspond to multiplication by some matrix).
(b) What are the conditions on a matrix $A$ which would guarantee that the motion of the plane defined by $A$ preserves distances (and, hence, is a rigid motion)? Treat this as an open-ended question: list the conditions you can come up with; convince the other members of your group that any "new" condition you suggest to put on the list is really new and necessary. If you think you have listed ALL possible conditions try to prove your list is sufficient.

