CENTER OF GRAVITY

LECTURE 5, EXERCISE SET 1

Recall that the group M os all rigid motions is generated by the following three "families" of elements:

(1) Translations $t_{\vec{a}}$ ($t_{\vec{a}}\vec{v} = \vec{v} + \vec{a}$)
(2) Rotations around the origin counterclockwise ρ_ϕ.
Rotation preserves the origin. and can be described by a rotation matrix $\rho_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$
(3) And just ONE reflection r - reflection through (around, over, under or with or without respect to) the x-axis. The reflection r also fixes the origin and correspond to the matrix $r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Any orientation-reversing rigid motion of the plane can be obtained by subsequent compositions of a reflection r followed by a rotation followed by a translation; for orientation-preserving motion skip the reflection.

Definition. Let $s_1, \ldots, s_n$ be n points on the plane. The *center of gravity* is the point whose coordinates are the ariphmetic means of the coordinates of s_i :

$$p = \frac{s_1 + \ldots + s_n}{n}$$

Exercise. Show that rigid motions preserve centers of gravity.

Hint: Since the group of all rigid motions is generated by translations, rotations and a reflection, it suffices to do the exercise for those three. So, here is a reformulation:

Exercise 1. Show that the following rigid motions preserve centers of gravity:
(1) Rotation ρ_ϕ,
(2) Translation $t_{\vec{a}}$,
(3) Reflection through the x-axis r.

Recall that we denote by M the group of all rigid motions of the plane. An (orthogonal) subgroup $\mathbb{O} < M$ is the subgroup of all motions which fix the origin. A subgroup $T < M$ is the subgroup of all translations of the plane.

Definition. A subgroup of M is called *discrete* if it does not contain arbitrarily small rotations or translations.

Exercise 2. Show that a discrete subgroup of M consisting of rotations around the origin is cyclic and is generated by some rotation ρ_θ.

Exercise 3. Show that a discrete subgroup of $\mathbb{O}$ is a finite group.

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