CENTER OF GRAVITY

SYMMETRY AND ESCHER, EXERCISE SET 8 (LECTURE 5)

The group M of all rigid motions is generated by the following three "families" of elements:

(1) Rotation ρ_ϕ CC around the origin, $0 \leq \phi < 2\pi$
Rotation preserves the origin.
Rotation ρ_ϕ is given by multiplication by the "rotation" matrix $\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

(2) Reflection r through the x-axis
The reflection r also fixes the origin and corresponds to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(3) Translations $t_{\vec{a}}$ ($t_{\vec{a}}\vec{v} = \vec{v} + \vec{a}$)

Any orientation-reversing rigid motion of the plane can be obtained as a composition of the reflection r followed by a rotation followed by a translation; for orientation-preserving motion skip the reflection.

Definition. Let $s_1, \ldots, s_n$ be n points on the plane. The *center of gravity* is the point whose coordinates are the arithmetic means of the coordinates of s_i :

$$p = \frac{s_1 + \ldots + s_n}{n}$$

Exercise. Show that rigid motions preserve centers of gravity.

Hint: Since the group of all rigid motions is generated by translations, rotations and a reflection, it suffices to do the exercise for those three. So, here is a reformulation:

Exercise 1. Show that the following rigid motions preserve centers of gravity:
(1) Rotation around the origin,
(2) Translation $t_{\vec{a}}$,
(3) Reflection through the x-axis.

Problem 2. "Fixed Point theorem". Let G be a finite group of rigid motions of the plane. Prove that G has a fixed point (that is, there is a point P on the plane which is fixed by all elements of G).

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The group M has two important subgroups:

(1) An (orthogonal) subgroup $\mathbb{O} < M$ which consists of all motions that fix the origin.
(2) A subgroup $T < M$ of all translations of the plane.

Definition. A subgroup of M is called *discrete* if it does not contain arbitrarily small rotations or translations.

Problem 3. Show that a discrete subgroup of M consisting of rotations around the origin is cyclic and is generated by some rotation ρ_θ.

Problem 4. Show that a discrete subgroup of $\mathbb{O}$ is a finite group.