POINT GROUPS

SYMMETRY AND ESCHER, EXERCISE SET 9 (LECTURE 6)

Recall that we denote by M the group of all rigid motions of the plane. An (orthogonal) subgroup $\mathbb{O} < M$ is the subgroup of all motions which fix the origin. A subgroup $T < M$ is the subgroup of all translations of the plane.

Definition. Let G be a discrete group of rigid motions of the plane. The *translation subgroup* of G is the subgroup generated by all translations $t_{\vec{a}}$ in G.

There are three possibilities: L is trivial (then G is a finite group of rigid motions), L is generated by just one translation (this leads to frieze patterns), and L is generated by two indepenent translations.

Definition. A discrete group of rigid motions of the plane is called a 2-dimensional **crystallographic group** if the subgroup L of G is a lattice, i.e., L is generated by two linearly independent vectors $\vec{a}$, $\vec{b}$.

There is one-to-one correspondence:

$$\text{Wallpaper patterns} \longleftrightarrow \text{Crystallographic groups}$$

We now prove the theorem known as *crystallographic restriction*.

Problem 1. Let $H < \mathbb{O}$ be a finite subgroup of the group of symmetries of a lattice L. Then

(a) Every rotation in H has order $1, 2, 3, 4$, or 6.
(b) H is one of the groups C_n or D_n for $n = 1, 2, 3, 4$, or 6.

If G is a crystallographic group, and L is the lattice of G, then $\overline{G} = G/L$ is the point group. The group $\overline{G}$ consists only of rotations and reflections, and is a finite subgroup of $\mathbb{O}$. It also carries the lattice L to itself. Hence, we have the following important corollary:

Corollary 2. *Let G be a 2-dimensional crystallographic group; that is, G is a group of symmetries of a wallpaper pattern. Then the choice for the point group of G (the group which "encodes" all rotations, reflections and glide reflections) is very limited: it is one of the nine (only!!) groups from the list in the Exercise 3b.*

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