

MATRICES AND LINEAR TRANSFORMATIONS

SYMMETRY AND ESCHER, EXERCISE SET 5 (LECTURE 3)

Exercise 1. In each example below, multiply the given matrices.

$$(1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$(2) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(3) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is called a *permutation* matrix. Why is that :)?

$$(4) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This is the end of the first exercise. When you are done, choose one person who may possibly go to the board to write down the answer to one of the exercises. Raise your hand when your entire team (that is, everyone at your table) is finished and a team representative has been determined (via a democratic procedure, preferably).

Recall a few definitions and facts about matrices:

Definition. The matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the *identity* matrix.

Definition. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The *determinant* of A is $\det A = ad - bc$.

Definition. A matrix A is called *invertible* if there exists a matrix B such that $AB = BA = I$.

Theorem 1. A matrix A is invertible if and only if $\det A \neq 0$.

Proof. This follows easily from Exercise 1(4). Think it through if you haven't seen this before! \square

Let $M_2(\mathbb{R})$ denote the set of all 2×2 matrices of real numbers.

Definition. (This is from the first lecture). The group (under matrix multiplication)

$$\mathrm{GL}_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) \mid \det A \neq 0\}$$

is called the *general linear group*. It consists of all invertible 2×2 matrices.

Exercise 2. Describe geometrically the transformations of the plane given by the following matrices (for example, the permutation matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a reflection through the line $x = y$).

$$(1) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$(2) \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$(3) \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(4) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$(5) \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$(6) \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

When you are done, answer the following questions:

- Which of the motions (1)-(6) are orientation preserving?
- Which are orientation reversing?
- Give a necessary and sufficient condition for a matrix A to determine an orientation-preserving motion
- Give a necessary and sufficient condition for a matrix A to determine a rigid motion (that is, to preserve distance)