CENTER OF GRAVITY AND THE FIXED POINT THEOREM

SYMMETRY AND ESCHER, PROBLEM SET 6 (LECTURE IV)

Recall that the group $\mathbb{M}$ of all rigid motions of the plane is generated by the following three "families" of elements:

(1) Rotation ρ_θ CC around the origin, $0 \leq \theta < 2\pi$
Rotation preserves the origin.
Rotation ρ_θ is given by multiplication by the "rotation" matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(2) Reflection r through the x-axis
The reflection r also fixes the origin and corresponds to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(3) Translation $t_{\vec{v}}$ by a vector $\vec{v} = (a, b)$ for $a, b \in \mathbb{R}$.

Definition. Let $s_1, \ldots, s_n$ be n points on the plane. The *center of gravity* is the point whose coordinates are the arithmetic means of the coordinates of s_i :

$$p = \frac{s_1 + \ldots + s_n}{n}$$

Problem 1. Prove that rigid motions preserve centers of gravity.

Hint: It suffices to prove this for ρ_θ, r and $t_{\vec{v}}$.

Problem 2. Now prove the "Fixed Point theorem".

Theorem 3. *Let G be a finite group of rigid motions of the plane. Prove that G has a fixed point (that is, there is a point P on the plane which is fixed by all elements of G).*

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