Name ______________________________ Quiz Section __________

MIDTERM I Solutions

Math 126, Section A
January 25, 2007

1. (6pts) Find the Taylor series for the function $f(x) = e^x$ based at $a = 1$.

   Answer. $f(x) = \sum_0^\infty e\frac{(x-1)^n}{n!}$, converges everywhere.

2. (12pts) Let $f(x) = \int_0^x \frac{\cos t - 1}{t^2} dt$.

   (a)(9pts) Find the Taylor series of $f(x)$ based at $a = 0$.

   Solution. First, we compute the Taylor series at 0 for the function under the integral.
   $\cos t - 1 = (1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots) - 1 = -\frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots$.
   $\frac{\cos t - 1}{t^2} = \frac{-\frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots}{t^2} = -\frac{1}{2} + \frac{t^2}{4!} - \frac{t^4}{6!} + \ldots = \sum_1^\infty \frac{(-1)^n x^{2n-2}}{(2n)!}$.
   Integrating, we get $f(x) = \int_0^x (-\frac{1}{2} + \frac{t^2}{4!} - \frac{t^4}{6!} + \ldots)dt = -\frac{x}{2} + \frac{x^3}{3 \cdot 4!} - \frac{x^5}{5 \cdot 6!} + \ldots = \sum_1^\infty (-1)^n \frac{x^{2n-1}}{(2n-1)(2n)!}$

   (b)(3pts) Find $f^{(5)}(0)$.

   Solution. We can read off $T_5(x)$ for $f(x)$ from the series above:

   $$T_5(x) = -\frac{x}{2} + \frac{x^3}{3 \cdot 4!} - \frac{x^5}{5 \cdot 6!}.$$

   Hence, the coefficient by x^5 is $-\frac{1}{5 \cdot 6!}$. On the other hand, by the general formula for Taylor polynomials, it must be $\frac{f^{(5)}(0)}{5!}$. Hence,

   $$\frac{f^{(5)}(0)}{5!} = -\frac{1}{5 \cdot 6!}$$

Solving for $f^{(5)}(0)$, we get $f^{(5)}(0) = -1/30$.

3. (15pts) Let $f(x) = \cos 2x$.

(a)(5pts) Find the Quadratic approximation for $f(x)$ at $a = 0$.

Answer. $T_2(x) = 1 - 2x^2$.

(b)(5pts) Find the error bound for the Quadratic approximation above on the interval $[-0.5, 0.5]$.

Solution. By the Taylor's inequality, we have $|f(x) - T_2(x)| \le \frac{M}{6}|x|^3$. M is the maximum of $f'''(x) = 8\sin(2x)$ on the interval $[-0.5, 0.5]$. Since $|\sin(2x)|$ is bounded by 1, we can take $M = 8$. Furthermore, $|x|$ is bounded by 0.5 on our interval. Hence, the error bound is $\frac{8}{6}(0.5)^3 = 1/6$.

Variation. Many students observed that we can be even more efficient finding M. Since $f'''(x) = 8\sin(2x)$ is increasing on the interval $[-0.5, 0.5]$, the maximum occurs at $x = 0.5$. Hence, $M = 8\sin(2 * 0.5) = 8\sin(1) \simeq 8 * 0.841 = 6.73$ (Be careful here: x is measured in RADIANS, not in degrees). With this M, the error bound becomes $\frac{6.73}{6}(0.5)^3 \simeq 0.14$. It was OK to leave your answer in the exact form: $\frac{\sin(1)}{6}$.

(c)(5pts) Find the n^{th} Taylor polynomial $T_n(x)$ of $f(x)$ based at $a = 0$ such that the error $|T_n(x) - \cos x|$ is at most 0.1 on the interval $[-0.5, 0.5]$. Justify your answer.

Solution. When doing the derivatives in (a) we can observe the pattern:

$$f^{(n+1)}(x) = \pm 2^{n+1}\sin(2x) \quad \text{if} \quad \text{n} \quad \text{is} \quad \text{even},$$

and

$$f^{(n+1)}(x) = \pm 2^{n+1}\cos(2x) \quad \text{if} \quad \text{n} \quad \text{is} \quad \text{odd}.$$

In either case, $|f^{(n+1)}(x)| \le 2^{n+1}$. Hence, in the Taylor's inequality

$$|f(x) - T_n(x)| \le \frac{M}{(n+1)!}|x|^{n+1}$$

we can take M to be 2^{n+1}, since M is the bound for $|f^{(n+1)}(x)|$. (It is very important to notice here that M DEPENDS on n. In particular, it is wrong to simply take M that we bound in (b) and use it here for all derivatives. M from (b) was specific for the third derivative of f.)

With $M = 2^{n+1}$, the Taylor's inequality becomes

$$|f(x) - T_n(x)| \le \frac{2^{n+1}}{(n+1)!}|x|^{n+1}$$

Since $|x|$ is at most 0.5 on our interval, we further simplify the bound to be

$$\frac{2^{n+1}}{(n+1)!}|0.5|^{n+1} = \frac{1}{(n+1)!}.$$

Hence, it suffices to find n such that $\frac{1}{(n+1)!} \leq 0.1$. By trial and error method we get that $n = 3$ works. Hence, $n \geq 3$.

Variation. Instead of figuring out the general formula for $f^{(n+1)}(x)$, we can apply the "trial-and-error" method from the very beginning. From (b) we know that the Taylor inequality for $n = 2$ gives the bound 0.14. Try $n = 3$. The calculation is very similar to the one we did in (b), but the error bound turns out to be smaller than 0.1. Hence, $n = 3$ works.

Comment. There was one student who observed that $n = 2$ works as well! Indeed, the Taylor polynomials for $\cos(2x)$ only have even powers in them. Hence, $T_3(x) = T_2(x)$. Since T_3 works, T_2 must work too.

4. (7pts) Check whether the points $(1, 2, 3)$, $(-2, 5, 7)$, and $(-5, 8, 11)$ lie on the same line.

Solution. Let $A = (1, 2, 3)$, $B = (-2, 5, 7)$, and $C = (-5, 8, 11)$. The points are on the same line if and only if the vectors AB and AC are colinear. We compute $AB = (-3, 3, 4)$, and $AC = (-6, 6, 8) = 2(-3, 3, 4)$. Since these vectors are proportional to each other, they lie on the same line. Therefore, so do the points A, B, and C.

Remark: This problem can also be solved by computing the angle between the vectors AB and AC, or by computing the area of the trianlge ABC via the cross product.

5. (9pts) Find the angle between the two main diagonals of a unit cube. (A unit cube is a cube with the vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$; the main diagonals are the diagonals connecting the vertex $(0, 0, 0)$ with the vertex $(1, 1, 1)$, and the vertex $(1, 0, 0)$ with the vertex $(0, 1, 1)$).

Solution. We just have to compute the angle between the diagonal vectors $\mathbf{u} = (\mathbf{1}, \mathbf{1}, \mathbf{1}) - (\mathbf{0}, \mathbf{0}, \mathbf{0}) = (\mathbf{1}, \mathbf{1}, \mathbf{1})$ and $\mathbf{v} = (\mathbf{0}, \mathbf{1}, \mathbf{1}) - (\mathbf{1}, \mathbf{0}, \mathbf{0}) = (-\mathbf{1}, \mathbf{1}.\mathbf{1})$.

We get $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = \frac{1}{3}$. Hence, $\theta = \arccos(\frac{1}{3})$.