MIDTERM I ANSWERS

Math 126, Section C

October 18, 2006

NOTE. Most problems had multiple solutions. Below we give only one possible solution for each problem. It is perfectly acceptable if you had a different (but correct!) solution.

1. Find the Taylor series for a given function $f(x)$. Give your answer using summation notation.

   (a) (7pts.) $f(x) = e^x$, based at $a = 2$

   Answer. $e^x = \sum_0^\infty \frac{e^2}{n!}(x-2)^n$

   (b) (8pts.) $f(x) = \ln(1-2x)$, based at $a = 0$.

   Answer. $f'(x) = -\frac{2}{1-2x} = -2\sum_0^\infty 2^n x^n$, hence,

   $f(x) = -\sum_0^\infty \frac{2^{n+1}}{n+1}x^{n+1} = -\sum_1^\infty \frac{2^n}{n}x^n$

2. (15pts.) Let $f(x) = \frac{1}{(1-x)(1+x)}$.

   (a) (7pts.) Find the Taylor series for $f(x)$ based at $a = 0$, and the interval of convergence. Give your answer using the summation notation.

   Answer. $f(x) = \frac{1}{(1-x)(1+x)} = \frac{1}{1-x^2} = \sum_0^\infty x^{2n}$. The interval of convergence is $(-1, 1)$.

   (b) (4pts.) Find the 6^{th} Taylor polynomial of $f(x)$ based at $a = 0$.

   Answer. $T_6(x) = 1 + x^2 + x^4 + x^6$

   (c) (4pts.) Find $f^{(6)}(0)$.

   Answer. $f^{(6)}(0) = 6!$

3. Let $f(x) = 2\cos^2 x - 1$.

   (a) (6pts.) Find the quadratic approximation $T_2(x)$ of $f(x)$ based at $a = 0$

   Answer. Using the double angle formula, we get $f(x) = \cos(2x)$. Hence, $f(x) = \sum_0^\infty (-1)^n \frac{2^{2n}x^{2n}}{(2n)!}$. Cutting off the tail, we obtain

$$T_2(x) = 1 - 2x^2$$

   (b) (3pts.) Use the quadratic approximation to estimate $f(\frac{\pi}{8})$.

   Answer.

$$T_2(\frac{\pi}{8}) = 1 - 2(\frac{\pi}{8})^2 = 1 - \frac{\pi^2}{32}$$

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(c) (6pts.) Using Taylor's inequality, find the error bound for the estimate you computed in (b).

Answer. $|f^{(3)}(x)| = |8\sin(2x)| \le 8$. Hence, using Taylor's inequality, we get

$$|f(\frac{\pi}{8}) - T_2(\frac{\pi}{8})| \le \frac{8}{6}(\frac{\pi}{8})^3 = \frac{\pi^3}{6 * 64} \sim \frac{9}{128}$$

4. (15pts.) Let $A = (3,0,0)$, $B = (0,4,0)$, and $C = (0,0,1)$.

(a) (8pts.) Find the area of the triangle ABC

Answer. 13/2

(b) (7pts.) Let CH be the height of the triangle from the vertex C to the base AB. Find the coordinates of the point H.

Answer. Let $\overrightarrow{u} = \overrightarrow{AB} = (-3,4,0)$, $\overrightarrow{v} = \overrightarrow{AC} = (-3,0,1)$. Then

$$\overrightarrow{AH} = proj_{\overrightarrow{u}}\overrightarrow{v} = \frac{\overrightarrow{v}\cdot\overrightarrow{u}}{||\overrightarrow{u}||^2}\overrightarrow{u} = \frac{(-3,4,0)\cdot(-3,0,1)}{3^2+4^2}(-3,4,0) = \frac{9}{25}(-3,4,0) = (-\frac{27}{25},\frac{36}{25},0)$$

Let O denote the origin. Then the coordinates of H are the same as of the vector $\overrightarrow{OH}$. We have

$$\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{AH} = (3,0,0) + (-\frac{27}{25},\frac{36}{25},0) = (\frac{48}{25},\frac{36}{25},0) = (1.92, 1.44, 0)$$