NAME ______________________ QUIZ SECTION ________

MIDTERM II Answers/Solutions

Math 126, Section C

November 17, 2006

1. (9pts) A curve is given by the equation

$$r + \frac{1}{r} = 4\cos\theta$$

in polar coordinates.

(a) (7pts) Find an equation of the curve in Cartesian coordinates. Sketch the curve.

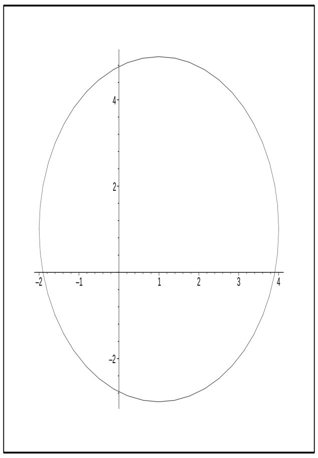
Solution.
Multiplying both sides by r, we get
$r^2 + 1 = 4r\cos\theta$
$x^2 + y^2 + 1 = 4x$
$(x-2)^2 + y^2 = 3$
Hence the curve is a circle with the center $(2, 0)$ and radius $\sqrt{3}$.

(b) (2pts) Using your sketch, find all the points on the curve where the tangent line is horizontal.

Solution. Since the curve is a circle, the tangent line is horizontal at the points right above and right below the center: $(2, \sqrt{3})$ and $(2, -\sqrt{3})$.

2. (15pts) Consider the ellipse given by the equation

$$\frac{(x-1)^2}{9} + \frac{(y-1)^2}{16} = 1$$



(a) (5pts) Find parametric equations of the ellipse.

Solution. $(\frac{x-1}{3})^2 + (\frac{y-1}{4})^2 = 1$. Hence, set $\frac{x-1}{3} = \cos t$, $\frac{y-1}{4} = \sin t$. Solving for x, y, we get

$$x = 3\cos t + 1, \quad y = 4\sin t + 1$$

(b) (5pts) Set up, but do not evaluate, the integral computing the length of the arc of the ellipse going clockwise from the point $(1,5)$ to the point $(4,1)$.

Answer. $\int_0^{\pi/2} \sqrt{9\sin^2(t) + 16\cos^2(t)}dt$

(c) (5pts) Using the sketch of the ellipse, find the points where the curvature of the ellipse is minimal. Compute the value of the curvature at these points.

Answer. The curvature is minimal at the most left and most right points of the ellipse: $(-2,1)$ and $(4,1)$, corresponding to $t = 0$ and $t = \pi$. The curvature at these two points is the same, so we shall compute κ at $t = 0$.
We shall use the formula

$$\kappa = \frac{||r' \times r''||}{||r'||^3} \qquad *$$

for the curvature. We compute

$$r'(t) = (-3\sin t, 4\cos t), \quad r''(t) = (-3\cos t, -4\sin t)$$

Hence, $r'(0) = (0,4)$, $r''(0) = (-3,0)$. We can in particular observe that $r'(0)$ and $r''(0)$ are perpendicular (take the dot product). Hence,

$$||r'(0) \times r''(0)|| = ||r'||||r''|| \sin(\pi/2) = 4 \cdot 3 \cdot 1 = 12$$

Since $||r'(0)|| = 4$, the formula (*) implies

$$\kappa = \frac{12}{4^3} = \frac{3}{16}$$

3. (12pts) A particle is moving with the acceleration $\vec{a}(t) = (-\cos t, -\sin t, 0)$, $t \geq 0$. The initial position of the particle is $(1, 0, 0)$ and the initial velocity is $(0, 1, 1)$

(a) (6pts) Find the position vector $\vec{r}(t)$.

Solution. $\vec{v}(t) = (-\sin t, \cos t, 1)$, $\vec{r}(t) = (\cos t, \sin t, t)$.

(b) (6pts) Find the length of the projection of the acceleration vector onto the unit tangent vector.

Solution. $T = \frac{\vec{r}'}{||\vec{r}'||} = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)$ Hence,

$a_T = comp_T\vec{a} = \frac{\vec{a}\cdot T}{||T||} = \vec{a} \cdot T = (-\cos t, -\sin t, 0) \cdot \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) = 0.$

Alternatively, we can use the formula $a_T = v'$. Here, $v = ||\vec{r}'(t)|| = \sqrt{2}$. Hence, $a_T = v' = 0$.

4. (12pts) Let $f(x, y) = \sqrt{11 - x^2 - y^2}$.

(a) (3pts) Sketch the contour map of the function f consisting of level curves for the level $k = 0, 1, 2, 3$. Label the curves.

(b) (3pts) Compute $f_x(x, y)$.

Answer. $f_x(x, y) = -\frac{x}{\sqrt{11-x^2-y^2}}$

(c) (3pts) Compute $f_y(x, y)$.

Answer. $f_y(x, y) = -\frac{y}{\sqrt{11-x^2-y^2}}$

(d) (3pts) Find an equation of the tangent plane to the graph of f at the point $(1, 1, 3)$.

Answer. $f_x(1, 1) = -1/3$, $f_y = -1/3$. Hence

$$z - 3 = -\frac{1}{3}(x - 1) - \frac{1}{3}(y - 1)$$

or

$$x + y + 3z = 11$$

5. (12pts) Consider a curve given by the parametric equations $(t, t^2 - 1, t^3 + 1)$.

(a) (8pts) Find an equation of the normal plane at the point $(0, -1, 1)$.

Solution. $r'(t) = (1, 2t, 3t^2)$. The point $(0, -1, 1)$ corresponds to $t = 0$. The normal vector to the normal plane is the tangent vector. To get the tanget vecotr at the point $(0, -1, 1)$ we have to plug in $\mathbf{t = 0}$ into $r'(t) = (1, 2t, 3t^2)$. (Caution: we do not plug in the point $(0, -1, 1)$ itself anywhere, we find the corresponding value of t, and then plug it in). We get $r'(0) = (1, 0, 0)$. The equation of the normal plane is

$$x = 0$$

(b) (4pts) Find the angle between the normal and osculating planes at the point $(1, 0, 2)$.

Solution. The normal vector to the normal plane is T, and to the osculating plane is B. Since T is perpendicular to B (always!), the angle betwen the two planes is $\pi/2$.

6(Bonus) (2pts) Let $\vec{r}(t) = (\cos t \sin t, \sin^2 t, \cos t)$. Show that $\vec{r}'$ is perpendicular to $\vec{r}$ at any point on the curve WITHOUT computing $\vec{r}'$.

Solution. Since $r'(t)$ is perpendicular to $r(t)$ whenever $r(t)$ has constant length, it suffices to show that the given vector function r the length is independent of t, i.e. a constant. We compute

$$||r(t)|| = \sqrt{\cos^2 t \sin^2 t + \sin^4 t + \cos^2 t} = \sqrt{\sin^2 t(\cos^2 t + \sin^2 t) + \cos^2 t} = \sqrt{\sin^2 t + \cos^2 t} = 1$$

We used the identity $\cos^2 t + \sin^2 t = 1$ twice in this calculation.