Sample MIDTERM II, version 1
Last year's midterm for Spring MATH 126 A, B

Scientific, but not graphing calculators are OK.

You may use one 8.5 by 11 sheet of handwritten notes.

Problem 1. Consider a particle traveling according to the equations

$$x(t) = \cos^2 t, \quad y(t) = \cos t.$$

Write down and simplify (but do not evaluate) the formula for the length of the curve along which the particle is moving.

Answer. Assuming that the particle starts moving at the time $t = 0$, we get

$$L = \int_0^{t_0} \sqrt{(x'(t))^2 + (y'(t))^2 dt} = \int_0^{t_0} \sqrt{\sin^2(2t) + \sin^2 t}$$

or

$$L = \int_0^{t_0} |\sin t| \sqrt{4\cos^2 t + 1}$$

Problem 2. Consider a particle whose velocity, at time $t \geq 0$, is given by

$$\vec{v}(t) = \langle\, -2t\,,\, -\sin t\,\rangle$$

and whose position at $t = 0$ is $(4, 0)$.

a. Find the formula for the position of the particle at time t.

b. Find the point at which the particle crosses the y axis.

c. Suppose the acceleration suddenly drops to 0 at the time when the particle crosses the y-axis, so that there are no forces acting on the particle. Find the position of the particle one minute later.

Answers.
a. $\vec{r}(t) = (-t^2 + 4, \cos(t) - 1)$
b. This happens at $t = 2$. The point is $(0, \cos(2) - 1) \simeq (0, -1.4)$
c. When the acceleration drops to zero, the particle continues to move in the direction it was moving, i.e. the velocity is constant. Hence, the position function for $t \geq 2$ is $\vec{r}(2) + (t-2)\vec{v}(2) = (0, \cos(2) - 1) + (t-2)(-4, -\sin(2)) = (-4(t-2), -t\sin(2) + \cos(2) + 2\sin(2) - 1) \simeq (-4(t-2), -0.9t + 0.4)$. At $t = 3$, we get $(-4, -2.3)$.

Problem 3. Find the equations of the <u>normal</u> and of the <u>osculating</u> planes to the curve

$$\vec{r}(t) = \langle\, t^3 \,,\, \sin(\pi t) \,,\, t+1 \,\rangle$$

at the point corresponding to $t = 2$.

Answers. $\vec{r}(2) = (8, 0, 3)$,
$\vec{r}'(t) = (3t^2, \pi\cos(\pi t), 1)$
$\vec{r}'(2) = (12, \pi, 1)$
Normal plane: $12(x-8) + \pi y + (z-3) = 0$

$\vec{r}''(t) = (6t, -\pi^2\sin(\pi t), 0)$
$\vec{r}''(2) = (12, 0, 0)$
$\vec{r}' \times \vec{r}'' = (0, 12, -12\pi)$ at the point $t = 2$. Hence the normal to the osculating plane can be taken to be $(0, 1, -\pi)$.
Osculating plane: $y = \pi(z-3)$

Problem 4. Identify the curve

$$r = 2\sin\theta + 2\cos\theta$$

by finding a Cartesian equation for the curve. Give a verbal description of what that curve is.

Answer.
$r^2 = 2r\sin\theta + 2r\cos\theta$.
$x^2 + y^2 = 2x + 2y$
$(x-1)^2 + (y-1)^2 = 2$
Hence this is a circle of radius $\sqrt{2}$ with the center $(1, 1)$.

Problem 5. Consider the function of two variables

$$f(x, y) = \sqrt{1 + x - y^2}.$$

a. Identify and sketch the domain of $f(x, y)$.
b. Find the partial derivatives $f_y(x, y)$ and $f_x(x, y)$.
c. Find the second partial derivative $f_{xy}(x, y)$.
d. Find an equation of the tangent plane at the point $(1, 1)$.

Answer.
a. The domain is $x \geq y^2 - 1$
b. $f_x = \frac{1}{2\sqrt{1+x-y^2}}$, $f_y = -\frac{y}{\sqrt{1+x-y^2}}$.
c. $f_{xy} = \frac{y}{2(1+x-y^2)^{3/2}}$.
d. At the point $(1, 1)$, $f_x = 1/2$, $f_y = -1$ and $f(1, 1) = 1$. Hence the equation is $z - 1 = 1/2(x - 1) - (y - 1)$. Simplifying, we get

$$x - 2y - 2z = 1$$