MATH 256, Fall 2004
PRACTICE Midterm **Name:** ____________________

1. Find general solutions of the ODE

   (a) $y' - 2y = e^{3t}$

   Answer. $e^{3t} + ce^{2t}$

   (b) $y' = y^2/t$

   Answer. $-\frac{1}{\ln|t|+c}$

   (c) $y' = y(y+1)/t$

   Answer. $\frac{y}{y+1} = ct$. Equivalently, $y = \frac{t}{c-t}$.

   (d) $y' = y - y^3$ (Bernoulli equation)

   Answer. $y = (1 + ce^{-2t})^{1/2}$.

2. Solve initial value problems. Determine the longest interval where solution is well-defined. Describe the behaviour of the solution when $t \to \infty$ when applicable (converges, diverges, bounded, unbounded)

   (a) $y' = t, \quad y(2) = 5$ Describe behaviour at ∞.

   Answer. $y(t) = t^2/2 + 3$, diverges.

   (b) $y'' + e^t y' + t^3 y = 0, \quad y(1) = 0, y'(1) = 0$

   Answer. $y(t) = 0$

   (c) $ty' + 2y = \sin t, \quad y(1) = \sin 1 - \cos 1 + 1$

   Describe behaviour at ∞.

   Answer. $y(t) = \frac{\sin t}{t^2} - \frac{\cos t}{t} + \frac{1}{t^2}$, converges to 0 at $+\infty$.

   (d)$y' = 2t/(y + t^2 y), \quad y(0) = -2$

   Answer. $y(t) = -\sqrt{2\ln(1+t^2) + 4}$

   (e) $y' = 2y - 5y^3, \quad y(0) = 1$ (Bernoulli equation)

   Answer. $y(t) = \frac{1}{\sqrt{2.5 - 1.5e^{-4t}}}$

3. Find a fundamental set of solutions, compute Wronskian, give a general solution and then solve the initial value problem, sketch the graph of the solution and describe behaviour at ∞.

   (a)$y'' + 5y' + 6y = 0, \quad y(0) = 2, y'(0) = 2$

   Answer.

   $< e^{-2t}, e^{-3t} >$,
   $W = -e^{5t}$,
   $y(t) = 8e^{-2t} - 6e^{-3t}$,
   converges to 0 at $+\infty$, diverges to $-\infty$ at $-\infty$.

(b)$y'' + 2y' + 6y = 0, \quad y(0) = 2, y'(0) = 2$

Answer.

$< e^{-t}\cos(\sqrt{5}t), e^{-t}\sin(\sqrt{5}t) >$,
$W = e^{2t}$,
$y(t) = 2e^{-t}\cos(\sqrt{5}t) + 4e^{-t}\sin(\sqrt{5}t)$,
converges to 0 at $+\infty$, diverges at $-\infty$.

4. Someone is buying a house anticipating to be able to pay $1200(1 + t/60)$ where t is the number of months since the loan was made. Assuming the interest rate to be 5%, compute the price the person can afford if he wishes to pay off the loan in 20 years.

   Answer.

   Let's measure t in years. Then the payment rate will become $1200(1+t/5)$. Let $r = 0.05$. Let $S(t)$ be the amount owed at time t.

   Equation: $S'(t) = rS(t) - 1200(1 + t/5)$ (= interest added - payment made).

   Rewrite as $S'(t) - rS(t) = -1200 - 240t$. This is a linear equation. To solve, multiply by the integration factor e^{-rt} and integrate

$$e^{-rt}S(t) = -\int(1200e^{-rt} + 240te^{-rt})dt = 1200\frac{e^{-rt}}{r} + 240t\frac{e^{-rt}}{r} + 240\frac{e^{-rt}}{r^2} + c$$

   Divide by e^{-rt}.

$$S(t) = \frac{1}{r}(1200 + 240t + 240/r) + ce^{rt}$$

   Plug in $r = 0.05 = 1/20$

$$S(t) = 20(6000 + 240t) + ce^{t/20}$$

   It is given that $S(20) = 0$. Plug in $t = 20$

$$0 = 20(6000 + 4800) + c * e$$

   Thus,

$$c = -216000/e$$

   We get

$$S(t) = 20(6000 + 240t) - 216000e^{t/20-1}$$

   Finally, by plugging in $t = 0$, we get the price that the person can afford to pay:

$$S(0) = 20 * 6000 - 216000/e \simeq 160000$$

5. It is being observed that the number of poorly designed ballots at the presidential elections that later have to be disqualified grows by 20% with every election (or 5% yearly if one assumes that ballots are made continuously between the elections). Assuming that the number of registered voters, and, thus, the total number of ballots, remains constant

over the years and that the number of faulty ballots in 2000 consituted 5% of all ballots, predict in what year the elections will come to a halt because of no qualified ballots to count. You may use $\ln 20 \simeq 3$.

Answer. Let $F(t)$ be the function which counts the number of badly designed ballots at the time t. Then,

$$F'(t) = 0.05F(t)$$

Solving for F, we get

$$F(t) = F_0 e^{0.05t},$$

where F_0 is the number of bad ballots in 2000 which is when we set up our timer. Let R be the total number of ballots. Then $F_0 = 0.05R$. Thus,

$$F(t) = 0.05Re^{0.05t} = \frac{Re^{t/20}}{20}$$

We would like to know when $F(t)$ becomes R. Thus, we have to solve for t:

$$\frac{Re^{t/20}}{20} = R$$

Solving for t, we get $t = 20\ln(20) \simeq 60$. Thus, the answer is 2060.

6. For the following equations, draw the phase line, determine equilibrium solutions, classify them as asymptotically stable, unstable or semistable. Sketch several graphs of solutions on the ty-plane. Find general solution of the equation.

   (a) $y' = -17(y-2)^2$

   Answer. $y = 2$, semistable. General solution $y(t) = 1/(17t + c) + 2$

   (b) $y' = (y-1)(y-3)$

   Answer. $y = 1$, asymptotically stable, $y = 3$, unstable. General solution $y(t) = \frac{3-ce^{2t}}{1-ce^{2t}}$

7. Approximate the solution of the initial value problem $y' = 2y - 1, \quad y(1) = 0$ at $t = 2$. Take $h = 0.25$. Solve the equation, compare your approximation with the exact value of the solution at $t = 2$ and estimate the error.

   Answer. Euler's method yeilds $y_4 \simeq -2.03$.

   Solution: $y(t) = \frac{1-e^{2t-2}}{2}$. $y(2) = -3.19$. Error is about 30%.

8. Is it possible that the (missing) picture represents integral curves for the equation

   (a) $y' = y^{1/3}$

   Answer. Yes. $\delta f/\delta y = 1/3y^{-2/3}$ - discontinuous at $y = 0$. thus, the Uniqueness theorem does not apply and one can have intersecting integral curves.

   (b) $y' = y^{4/3}$

   Answer. No. Here $\delta f/\delta y = 4/3y^{1/3}$ is continuous. Thus, the uniqueness theorem implies that there is only one integral curve at any point: no intersections.

(c) $t^2y'' = y$

Answer. Yes. In the equation $y'' - 1/t^2y = 0$ the coefficient of y is undefined at $t = 0$. Thus, the uniqueness theorem does not apply.

(d) $y'' = t^2y$

Answer. Yes. Even though uniqueness theorem applies here (coefficients are continuous everywhere), the curves on the picture have different slopes at their intersection point. Thus, they can be solutions of different initial value problems involving this equation. (For an initial value problem of a second order equation we have to specify both $y(t_0)$ and $y'(t_0)$).

9. State the longest interval where the solution of the initial value problem

$$(t+1)y'' + \frac{1}{t-1}y' + (\ln t)\, y = 0, \quad y(1/2) = 0, y'(1/2) = 1$$

exists. Is solution unique? Explain.

Answer. (0.1). The solution is unique because this is a linear equation with coefficients continuous on the interval $(0, 1)$ containing the initial point $t = 1/2$

10. Does the initial value problem

$$y' = t^2 + y^2, \quad y(1) = 2, y'(1) = 239$$

have unique solution?

Answer. Yes. Both $f(t, y) = t^2 + y^2$ and $\delta f/\delta y = 2y$ are continuous functions, therefore, uniqueness theorem applies.