WORKSHEET ON SYMMETRIC GROUPS

DUE FRIDAY, OCTOBER 23

1. Generators of $S_n$

Definition 1.1. Symmetric groups on $n$ elements, denoted $S_n$, is a group of self-bijections (or permutations) of the set $X = \{1, 2, \ldots , n\}$, with multiplication defined by the composition.

Notation. Let $\sigma \in S_n$. Hence, $\sigma : \{1, 2, \ldots , n\} \to \{1, 2, \ldots , n\}$ is a bijection. The commonly used notation for the corresponding permutation is the following:

$$
\begin{pmatrix}
1 & 2 & \ldots & n \\
\sigma(1) & \sigma(2) & \ldots & \sigma(n)
\end{pmatrix}
$$

Definition 1.2. A permutation $\sigma \in S_n$ is called a cycle if there exists a subset $\{x_1, \ldots , x_k\} \subset \{1, 2, \ldots , n\}$ such that $\sigma(x_i) = x_{i+1}$ (assuming $k + 1 = 1$), and $\sigma(y) = y$ for any $y \neq x_i$. The standard notation for such a permutation is $(x_1, x_2, \ldots , x_k)$.

Two cycles $(x_1, x_2, \ldots , x_k)$ and $(y_1, y_2, \ldots , y_\ell)$ are called disjoint if the sets $\{x_1, x_2, \ldots , x_k\}$ and $\{y_1, y_2, \ldots , y_\ell\}$ do not intersect.

Example 1.3.

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 3 & 4 & 2 & 5
\end{pmatrix} = (234)
$$

Proposition 1.4. Any permutation $\sigma \in S_n$ can be written as a composition of disjoint cycles.

Remark 1.5. Such decomposition is unique up to the order of the factors.

Example 1.6.

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
5 & 3 & 4 & 2 & 1
\end{pmatrix} = (15)(234)
$$

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 1 & 3 & 5 & 4
\end{pmatrix} = (12)(3)(45) = (12)(45)
$$

A cycle of length 1, such as (3) in the example above, just indicates that the corresponding element is fixed under the permutation. These are often skipped when permutation is written as a product of cycles.

We now describe the conjugacy classes of $S_n$ ( = the orbits under the action by conjugation of $S_n$ on itself).

Theorem 1.7. Let $\sigma, \tau \in S_n$. Then $\sigma$ and $\tau$ are conjugate if and only if their decompositions into disjoint cycles can be put into one-to-one correspondence such that the corresponding cycles are of the same length.

In particular, the conjugacy class of a single cycle consists of all cycles of the same length.
Remark 1.8. The group $S_n$ is non-commutative for $n \geq 3$. Nonetheless, disjoint cycles always commute.

Definition 1.9. A transposition is a cycle of length 2.

Proposition 1.10. (Problem 3) The symmetric group $S_n$ is generated by transpositions.

2. Alternating group

Note that the symmetric group $S_n$ acts on polynomials on $n$ variables. Namely, we define $$(\sigma f)(x_1, \ldots, x_n) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$$

In short, $\sigma f = f \circ \sigma^{-1}$. For example, for $n = 3$, $\sigma = (12)$ a cycle of length 2, 
$$\sigma(x_1^2x_2^5) = x_2^2x_1x_3^5 = x_1x_2^2x_3^5.$$ 

For $\sigma = (123)$,
$$\sigma(x_1^2x_2^5) = x_3^2x_1x_2^5 = x_1x_2^5x_3^2.$$

Let
$$f(x_1, \ldots, x_n) = \prod_{i<j}(x_i - x_j).$$

Question. Do you know for which matrix $f(x_1, \ldots, x_n)$ is the determinant?

Note that for any $\sigma \in S_n$, we have $\sigma f = \pm f$. Define a map
$$\text{Sgn} : S_n \to \mathbb{Z}/2\mathbb{Z}$$
via $\text{Sgn}(\sigma) = -1$ if $\sigma f = -f$ and $\text{Sgn}(\sigma) = 1$ otherwise.

Proposition 2.1. (Problem 4) $\text{Sgn}$ is a group homomorphism.

Definition 2.2. A permutation $\sigma \in S_n$ is called even if $\text{Sgn}(\sigma) = 1$. Otherwise, it is called odd.

Corollary 2.3. The subset of all even permutations is a normal subgroup of $S_n$.

Definition 2.4. The subgroup of even permutations is called an alternating group $A_n$.

As we shall see in the following theorem, the sign of a permutation can be determined from its decomposition into transpositions.

Theorem 2.5. (Problem 5) (1) If $\tau \in S_n$ is a transposition, then $\text{Sgn}(\tau) = -1$
(2) A permutation $\sigma$ is even if and only if it can be written as a product of even number of transpositions.

We now determine generators of $A_n$.

Theorem 2.6. (Problem 6) The group $A_n$ is generated by 3-cycles of the form $(12i)$, $3 \leq i \leq n$. 