Optional problem set I, 506 Spring 2009

(will not be "officially" graded but feel free to turn in if you would like for your work to be checked)

R is a commutative ring with an identity element.

Problem 1. Prove a special case of the Nakayama lemma for which the "finitely generated" hypothesis for the module can be dropped:
Let R be a ring, M be an R-module, and $\mathfrak{a}$ be a nilpotent ideal such that $M = \mathfrak{a}M$. Show that $M = 0$. (An ideal $\mathfrak{a}$ is *nilpotent* if $\mathfrak{a}^n = 0$ for some $n \in \mathbb{N}$).

Problem 2. An R-module is called *projective* if it is a direct summand of a free module (This is only one of several equivalent definitions. We should see more in class but this one serves best for this problem). Show that if R is a local ring and M is a finitely generated projective module, then M is free.

Problem 3. Let R be a local Noetherian ring, and $\mathfrak{M}$ be the maximal ideal of R. Show that $\bigcap_{n=1}^{\infty} \mathfrak{M}^n = 0$.