## Midterm for 506, Spring 2009

Monday, May 4

## Problem 2.

(1) Let $k$ be an algebraically closed field. Describe $\operatorname{Spec} k\left[t, \frac{1}{t}\right]$. (Pictures are appreciated.)
(2) Let $k$ be an algebraically closed field of characteristic $p$. Describe Spec $\frac{k[t]}{t^{p}-1}$.

Solution. [1]. We have an embedding $k[t] \subset k\left[t, t^{-1}\right]$. If $\tilde{\mathfrak{p}}$ is a prime in $k\left[t, t^{-1}\right]$, then $\mathfrak{p}=\tilde{\mathfrak{p}} \cap k[t]$ is a prime in $k[t]$ (since restrictions of prime ideals are always prime). Then it is very easy to check that

$$
\tilde{\mathfrak{p}}=\left\{\left.\frac{r}{t^{i}} \right\rvert\, r \in \mathfrak{p}, i \in \mathbb{N}_{0}\right\}=\mathfrak{p}\left[\frac{1}{t}\right]
$$

Hence, prime ideals in $k\left[t, t^{-1}\right]$ are of the form $\mathfrak{p}\left[\frac{1}{t}\right]$ where $\mathfrak{p}$ is a prime in $k[t]$. Since all primes in $k[t]$ are principal ideals generated by linear polynomials, we further conclude that primes in $k\left[t, t^{-1}\right]$ are of the form

$$
\left\{\left.\frac{(t-a) f(t)}{t^{i}} \right\rvert\, a \in k, f(t) \in k[t], i \geq 0\right\}
$$

Which ones of these are prime? All that are proper. Which ones are proper? All except for the one corresponding to $a=0$. Indeed, if $a=0$, then

$$
\left\{\left.\frac{t f(t)}{t^{i}} \right\rvert\, a \in k, f(t) \in k[t], i \geq 0\right\}=\left\{\left.\frac{f(t)}{t^{i-1}} \right\rvert\, a \in k, f(t) \in k[t], i \geq 0\right\}=k\left[t, t^{-1}\right]
$$

If $a \neq 0$, then the equation $(t-a) f(t)=t^{i}$ does not have solutions since the LHS vanishes at $a$ and the RHS does not. Hence, for $a \neq 0$, the ideal does not contain 1 and is proper.

Hence, Spec $k\left[t, t^{-1}\right]=\mathbb{A}^{1}-\{0\}$.
[2]. We have a surjection $k[t] \longrightarrow k[t] /\left(t^{p}-1\right)$. Hence, prime ideals in $k[t] /\left(t^{p}-\right.$ 1) are in 1-1 correspondence with prime ideals in $k[t]$ which contain $\left(t^{p}-1\right)$. Prime ideals of $k[t]$ are principal ideals generated by $(t-a)$. Since char $\mathrm{k}=\mathrm{p}$, we have $t^{p}-1=(t-1)^{p}$. Hence, it is divisible by a unique linear polynomial, $t-1$. Therefore, there is only one prime ideal containing $t^{p}-1$, the ideal $\langle t-1\rangle$. Therefore, Spec $k[t] /\left(t^{p}-1\right)$ consists of one point (which can be identified with the point 1 on $\mathbb{A}^{1}$ ).
Problem 4. Let $X, Y$ be algebraic sets. Prove that $\phi: X \rightarrow Y$ induces an isomorphism between $X$ and a closed subset of $Y$ if and only if $\phi^{*}: k[Y] \rightarrow k[X]$ is surjective.

Lemma. Let $V$ be an algebraic set, and $W \subset V$ be a closed subset. Then $k[V] \rightarrow k[W]$ is onto.

Proof. Let $V, W$ be closed subsets of $\mathbb{A}^{n}$. Since $W \subset V$, we have $I(V) \subset I(W)$. Therefore, $k[V]=k\left[x_{1}, \ldots, x_{n}\right] / I(V) \longrightarrow k\left[x_{1}, \ldots, x_{n}\right] / I(W)=k[W]$ is onto.

Solution. Let $k[Y]=k\left[y_{1}, \ldots, y_{n}\right] / I(Y), k[X]=k\left[x_{1}, \ldots, x_{m}\right] / I(X)$. There is 1-1 correspondence between ideals of $k[Y]$ and ideals of $k\left[y_{1}, \ldots, y_{n}\right]$ containing
 pull-back of $\operatorname{Ker} \phi^{*}$ to $\left.k\left[y_{1}, \ldots, y_{n}\right]\right)$. Let $Z=V\left(\widetilde{\operatorname{Ker} \phi^{*}}\right)$. Then
(1) $Z$ is a closed subset of $Y$ since $I(Y) \subset \widetilde{\operatorname{Ker} \phi^{*}}$ by construction
(2) $k[Z]=k\left[y_{1}, \ldots, y_{n}\right] / \widetilde{\text { Ker } \phi^{*}} \simeq k[Y] / \operatorname{Ker} \phi^{*}$

The ring homomorphism $\phi^{*}$ factors as a composition of a surjection followed by injection:

$$
\phi^{*}: k[Y] \xrightarrow{f^{*}} k[Z]=k[Y] / \operatorname{Ker} \phi^{*} \xrightarrow{g^{*}} k[X]
$$

Hence, the map $\phi$ factors as

$$
\phi: X \xrightarrow{g} Z \xrightarrow{f} Y
$$

Since $Z \subset Y$ is a closed subset (where the embedding if given by the map $f$ ), we have $\operatorname{Im} \phi=\operatorname{Im} g$. We would like to show that $Z=\operatorname{Im} g$. Suppose not. Let $\operatorname{Im} g=$ $Z^{\prime} \subset Z$. Then $\phi^{*}$ further factors as $\phi^{*}: k[Y] \longrightarrow k[Z] \longrightarrow k\left[Z^{\prime}\right] \longrightarrow k[X]$. The map $k[Y] \rightarrow k\left[Z^{\prime}\right]$ is onto by the Lemma and has the same kernel as the map $k[Y] \rightarrow k[Z]$ since both kernels are $\operatorname{Ker} \phi^{*}$. Hence, $k[Z] \rightarrow k\left[Z^{\prime}\right]$ is a surjective map without a kernel. It must be an isomorphism! We get the desired result: $Z=Z^{\prime}$.

We have shown $Z=\operatorname{Im} g=\operatorname{Im} \phi$. Suppose $\phi$ induces an isomorphism $\phi: X \simeq$ $\operatorname{Im} \phi=Z$. Then $g: X \rightarrow Z$ is an iso, and therefore $g^{*}: k[Z] \rightarrow k[X]$ is an iso. Since $f^{*}$ is surjective, we conclude that $\phi^{*}$ is onto.

Now suppose $\phi^{*}$ is onto. Then $k[Y] / \operatorname{Ker} \phi^{*} \simeq k[X]$. Therefore, $g^{*}: k[Z] \rightarrow k[X]$ is an iso which implies that $g: X \rightarrow Z=\operatorname{Im} \phi$ is an iso.

Problem 5. Let $A$ be a principal ideal domain, and $M, N$ be $A$-modules. Find necessary and sufficient conditions for $M \otimes_{A} N$ to be zero.

Solution. I have to admit that I underestimated the difficulty of this problem. At the moment, it appears open-ended to me; if you think more about it and come up with a good answer, definitely let me know. My intention was to give 8 or 9 points out of 10 for the solution for finitely generated modules; so here is the 8 point worth solution:
Answer. Assume $M, N$ are finitely generated. Then $M \otimes N=0$ iff $\operatorname{Ann}(M)+$ $\operatorname{Ann}(N)=(1)$.
Proof. Since $A$ is a PID, $M$ decomposes (uniquely!) as $M=M_{f r}+\underset{p \text { inA }}{\bigoplus} M_{(p)}$, where $M_{f r}$ is a free module and $M_{(p)}$ is the $p$-torsion of $M$ for a prime $p \in A$; that is, the collection of all elements of $M$ annihilated by some power of $p$. An analogous decomposition holds for $N$. If $M_{f r} \neq 0$, then $M \otimes N=M_{f r} \otimes N+\ldots=$ $A^{\oplus m} \otimes N+\ldots=N^{\oplus m}+\ldots \neq 0$. Hence, $M \otimes N=0$ if and only if both $M$ and $N$ are torsion. Since $\otimes$ commutes with direct sums, we get

$$
M \otimes N=\left[\bigoplus_{p \in A} M_{(p)}\right] \otimes\left[\bigoplus_{p \in A} N_{(p)}\right]=\bigoplus_{p, q \in A} M_{(p)} \otimes N_{(q)}
$$

By the structure theorem for modules over PID, $M_{(p)} \simeq \underset{m_{i}>0}{\bigoplus} A /\left(p^{m_{i}}\right)$. Arguing as in the homework for $A=\mathbb{Z}$, we get the following

$$
\begin{aligned}
& A /\left(p^{i}\right) \otimes A /\left(q^{j}\right)=0 \text { if } p \neq q \\
& A /\left(p^{i}\right) \otimes A /\left(p^{j}\right)=A /\left(p^{\min (\mathrm{i}, \mathrm{j})}\right)
\end{aligned}
$$

Hence, if $M \otimes N=0$ then $M, N$ do not have $p$-torsion for the same $p$. Therefore, if a prime $p \in A$ annihilates an element in $M$ then it does not annihilate anything in $N$ (equivalently, $N_{(p)}=0$ ). Hence, $\operatorname{Ann}(M)$ and $\operatorname{Ann}(N)$ are relatively prime. Note, that since $M$ is finitely generated and torsion, $\operatorname{Ann}(M)$ is a non-zero ideal.

Conversely, if $\operatorname{Ann}(M)$ and $\operatorname{Ann}(N)$ are relatively prime then $N, M$ are torsion but they do not have $p$-torsion for the same $p$. Therefore, $M \otimes N=0$.

Comment. With a bit more work (using what we proved in class about any zero tensor being zero inside some finitely generated submodule), this proof works for infinitely generated modules IF we assume that $\operatorname{Ann}(M), \operatorname{Ann}(N) \neq 0$. Also, if annihilators are zero but both modules are torsion-free, then the tensor product is non-zero. The problematic case is the one when one of the modules has a lot of torsion, that is, if there are infinitely many $p \in A$ for which there exists $m \in M$ such that $p m=0$ (in which case Ann $\mathrm{M}=0$ but not at all due to the lack of torsion elements).

