Homework 1 for 506, Spring 2016
due Wednesday, April 13

R is a commutative ring with 1 unless specified otherwise.

The following problem should remind you of the Gauss lemma.
Problem 1. Let $R[x]$ be a polynomial ring with coefficients in R, and let

$$f(x) = a_n x^n + \cdots + a_0 \in R[x].$$

Prove the following:

(1) f is a unit in $R[x]$ if and only if a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotent;
(2) f is nilpotent if and only if $a_0, \ldots, a_n$ are nilpotent.

Problem 2. Show that in $R[x]$ the nilradical coincides with the Jacobson radical.

Defn. A prime ideal $\mathfrak{p} \in R$ is *minimal* if it does not contain any proper prime ideals.

Example. If R is an integral domain then the ONLY minimal prime is 0-ideal.

Problem 3. Let R be a Noetherian ring. Show that there is one-to-one correspondence between irreducible components of $\operatorname{Spec} R$ and minimal prime ideals of R.
Note that this implies that a Noetherian ring has finitely many minimal prime ideals.

Problem 4. Show that for a non-empty topological space X to be irreducible is equivalent to any of the following

(1) Any non-empty open subset is dense;
(2) Any two non-empty open subsets have a non-empty intersection.

Problem 5. Let X be a topological space. Prove the following:

(1) If $Y \subset X$ is irreducible, then $\overline{Y}$ is irreducible;
(2) Any irreducible subset is contained in a maximal irreducible subset.

Problem 6. Describe $\operatorname{Spec} R$ for

(1) $R = \mathbb{R}[x]/(x^n)$ for some $n > 0$
(2) $R = \mathbb{Z}[x]$ (here, just describe the points of this space)

Creative pictures are appreciated!