WORKSHEET ON ARTINIAN RINGS

DUE FRIDAY, APRIL 22, BY 1PM IN JOSH'S MAILBOX

All rings are commutative with 1. This worksheet pursues two main results on Artinian rings:

(1) An Artinian ring is a Noetherian ring of dimension 0 (Thm 9).
(2) Structure theorem for Artinian rings (Thm 12)

Definition 1. An ideal I is nilpotent if $I^n = \{x_1 \cdots x_n \,|\, x_i \in I\} = (0)$ for some $n \in \mathbb{Z}$.

Lemma 2. *Let R be a commutative Artinian ring. Show that $J(R)$ is a nilpotent ideal.*

Proof. □

This result holds for non-commutative rings. It is likely that your proof would work without change, but you'd have to be more careful about multiplying from the left or from the right. We will skip all this and assume that the result of Problem 4 holds for not necessarily commutative rings.

Theorem 3 (Hopkins-Levitzki theorem)**.** (1) *Let R be an Artinian ring (not necessarily commutative), and M be a finitely generated R-module. Prove that M is a Noetherian R-module.*
(2) *Conclude that an Artinian ring is Noetherian.*

Proof. Exercise □

Lemma 4. *Let R be an Artinian integral domain. Then R is a field.*

Proof. Exercise □

Proposition 5. *Let R be an Artinian ring. Then any prime ideal is maximal.*

Proof. Exercise □

Corollary 6. *Let R be an Artinian ring. Then the Krull dimension of R is zero.*

Proof. Exercise (one line proof though). □

Proposition 7. *Let R be an Noetherian ring. Then $\mathfrak{N}(R)$ is a nilpotent ideal.*

Proof. Exercise □

Lemma 8. *(1). Let $\mathfrak{p}$ be a prime ideal in R. Then* $\mathrm{rad}(\mathfrak{p}^n) = \mathfrak{p}$.
(2). Let $\mathfrak{p}_1, \mathfrak{p}_2$ be prime ideals in R which are also relatively prime. Then $\mathfrak{p}_1^n, \mathfrak{p}_2^m$ are relatively prime for any $n, m > 0$.

Proof. Exercise (use properties of radicals from class or from [AM] for a very short proof of 2)). □

Theorem 9. *A ring R is Artinian if and only if it is Noetherian of Krull dimension* 0.

Proof. Artinian implies Noetherian by Hopkins-Levitzki theorem; dimension is 0 by Cor. 6.

Now let R be a zero-dimensional Noetherian ring. By Problem 3 from Homework 1, R has finitely many minimal prime ideals; since dimension is zero, all prime ideals are maximal. Let $\{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$ be the set of all maximal ideals in R. Then $\mathfrak{N} = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n = \mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_n$. Hence, $\mathfrak{m}_1^\ell \cdot \ldots \cdot \mathfrak{m}_n^\ell = 0$ for a big enough ℓ by Prop. 7. Now show that R has a composition series and conclude that it is Artinian.
Finish the proof

□

Corollary 10. *Let R be a Noetherian local ring with a maximal ideal $\mathfrak{m}$. Then one of the following holds:*
(•) *either $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for any $n > 0$*
(•) *or there exists n such that $\mathfrak{m}^n = 0$. In the latter case, R is Artinian.*

Proof. Exercise □

In other words, a local Noetherian ring is Artinian if and only if the unique maximal ideal is nilpotent.

Lemma 11. *Let R_1, R_2 be Artinian rings. Then $R_1 \times R_2$ is also Artinian.*

Proof. Exercise □

Theorem 12. *Any Artinian ring decomposes uniquely (up to isomorphism) as a direct product of finitely many local Artinian rings.*

Proof. Exercise (don't forget Chinese Remainder theorem). □

Remark 13. For an Artinian ring R, $\operatorname{Spec} R$ is just a union of finitely many points. Zariski topology becomes a discrete topology. $\operatorname{Spec} R$ is irreducible if and only if R is local.