**Homework 3 for 506, Spring 2016**
due Friday, April 29

$R$ is a commutative ring with 1 unless specified otherwise.

**Problem 1.** Let $X = \operatorname{Spec} R$. Show that the principal open set $X_f$ is quasi-compact. (There is a hint in [AM], Ch. I, Ex. 17. )

**Problem 2.** Let $k = \mathbb{F}_q$ be a field of $q$ elements. Give an example of $f \in k[x, y]$ such that for *every* $\alpha \in k$, the ring $A = k[x, y]/f$ is not finitely generated as a module over $k[X - \alpha Y]$.

In other words, the proof we gave for Noether Normalization Lemma really fails for finite fields.

**Problem 3.** Let $n$ be a square-free integer, and let $k = \mathbb{Q}(\sqrt{n})$. Show that $a+b\sqrt{n}$ is integral over $\mathbb{Z}$ if and only if either $a, b \in \mathbb{Z}$ or $n \equiv 1 \bmod 4$ and $a, b \in \frac{1}{2}\mathbb{Z}$.

**Problem 4.** Let $R \subset S$ be a finite ring extension and let $\mathfrak{p} \subset R$ be a prime ideal. Prove that there exists a prime ideal $\mathfrak{q} \in S$ such that $\mathfrak{q} \cap R = \mathfrak{p}$.

In this case, we say that $\mathfrak{q}$ lies over $\mathfrak{p}$. We'll prove later that there are only finitely many ideals $\mathfrak{q}$ "lying over" $\mathfrak{p}$.