Homework 5 for 506, Spring 2016:
due Friday, May 20

A is a commutative ring with identity.

Problem 1. Let $\mathfrak{a} \subset A$ be an ideal, and M be an A-module. Show that

$$A/\mathfrak{a} \otimes_A M \simeq M/\mathfrak{a}M.$$

Problem 2. Prove that the Hom-functor is left exact, and even more:

(1) Show that $M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of A-modules if and only if for any A-module N,

$$0 \longrightarrow \mathrm{Hom}(M'', N) \longrightarrow \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(M', N)$$

is exact;

(2) Show that $0 \longrightarrow N' \longrightarrow N \longrightarrow N''$ is a an exact sequence of A-modules if and only if for any A-module M

$$0 \longrightarrow \mathrm{Hom}(M, N') \longrightarrow \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(M, N'')$$

is exact.

For both cases give examples showing that Hom is not exact.

Problem 3. Properties of tensor product. Let M, N, P be A-modules. Show that there are canonical isomorphisms (tensor product is over A):

(1) $M \otimes N \cong N \otimes M$
(2) $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$
(3) $(M \oplus N) \otimes P \cong M \otimes P \oplus N \otimes P$
(4) $A \otimes M \cong M$

Problem 4. Calculate:

(1) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$
(2) $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$

Problem 5.
(1). Let $f \in A$, and let $X = \mathrm{Spec}\, A$. The canonical homomorphism $\phi : A \to A_f$ induces a continuous map $\phi^* : \mathrm{Spec}\, A_f \to X = \mathrm{Spec}\, A$. Show that ϕ^* induces a homeomorphism between $\mathrm{Spec}\, A_f$ and the principal open set X_f.

(2). Let $\mathfrak{p}$ be a prime ideal in A. Show that the canonical image of $\mathrm{Spec}\, A_{\mathfrak{p}}$ in $\mathrm{Spec}\, A$ is the intersection of all open subsets containing $\mathfrak{p}$.

Problem 6. Show that the following are equivalent for an A-module M:

(1) $M = 0$
(2) $M_{\mathfrak{p}} = 0$ for any prime ideal $\mathfrak{p}$ in A
(3) $M_{\mathfrak{m}} = 0$ for any maximal ideal $\mathfrak{m}$ in A