Homework 9 for 506, Spring 2016 due Wednesday, June 21

Throughout this homework, A is a commutative ring with identity.

Problem 1. Snake lemma. Assume that in the following commutative diagram of *A*-modules, the rows are exact:

$$\begin{array}{cccc} M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \\ & & \downarrow^{f} & \downarrow^{g} & \downarrow^{h} \\ 0 \longrightarrow N' \longrightarrow N \longrightarrow N''. \end{array}$$

Show that there is an exact sequence:

 $\operatorname{Ker} f \longrightarrow \operatorname{Ker} g \longrightarrow \operatorname{Ker} h \longrightarrow \operatorname{Coker} f \longrightarrow \operatorname{Coker} g \longrightarrow \operatorname{Coker} h$

Problem 2. Show that the following are equivalent:

- (1) P is a projective A-module;
- (2) Any surjective A-module homomorphism $M \to P$ splits;
- (3) $\operatorname{Hom}_A(P, -)$ is an exact functor.

Problem 3. Let *I* be an *A*-module. Prove that the following are equivalent:

(1) For any injective homomorphism $i: M' \to M$ and any homomorphism $g: M' \to I$ there exists $h: M \to I$ such that the following diagram commutes:

$$0 \longrightarrow M' \xrightarrow{i} M$$

$$f \downarrow \qquad f \downarrow \qquad$$

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- (2) The functor $\operatorname{Hom}_A(-, I) : \underline{A \operatorname{mod}} \to \underline{A \operatorname{mod}}$ is exact
- (3) Any exact sequence $0 \to I \longrightarrow M \to M'' \to 0$ splits

Hint: to prove 3) implies 1), take any diagram as in 1) and consider the module

$$I \oplus_{M'} M \stackrel{def}{=} \frac{I \oplus M}{\{(g(m'), -i(m')) \mid m' \in M'\}}$$

called the push-out of the diagram *:

$$\begin{array}{c|c} M' & \xrightarrow{i} & M \\ f & \downarrow & \downarrow \\ I & \longrightarrow I \oplus_{M'} M \end{array}$$

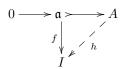
Show that the bottom horizontal map (induced by i) is injective; then apply 3) to that map.

Definition. A module satisfying one of these conditions is called injective.

In the next problem we shall describe injective modules over \mathbb{Z} . Note that this is a bit more involved than describing projective modules which are just \mathbb{Z}^n .

Problem 4.

I. Prove the **Baer's criterion** for injective modules: An A-module I is injective if and only if for any ideal $\mathfrak{a} \subset A$ and any map $f : \mathfrak{a} \to I$, the map f can be extended to $h : A \to I$:



II. Show that an abelian group is injective (as a \mathbb{Z} -module) if an only if it is divisible. (An abelian group A is divisible if for any $a \in A$, and any $n \in \mathbb{Z}$ there exists $b \in A$ such that a = nb. For example, \mathbb{Q} or \mathbb{Q}/\mathbb{Z} are divisible.)

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