Homework 9 for 506, Spring 2016
due Wednesday, June 21

Throughout this homework, A is a commutative ring with identity.

Problem 1. Snake lemma. Assume that in the following commutative diagram of A-modules, the rows are exact:

$$\begin{tikzcd}
 & M' \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, "g"] & M'' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N''. &
\end{tikzcd}$$

Show that there is an exact sequence:

$$\operatorname{Ker} f \longrightarrow \operatorname{Ker} g \longrightarrow \operatorname{Ker} h \longrightarrow \operatorname{Coker} f \longrightarrow \operatorname{Coker} g \longrightarrow \operatorname{Coker} h$$

Problem 2. Show that the following are equivalent:

(1) P is a projective A-module;
(2) Any surjective A-module homomorphism $M \to P$ splits;
(3) $\operatorname{Hom}_A(P, -)$ is an exact functor.

Problem 3. Let I be an A-module. Prove that the following are equivalent:

(1) For any injective homomorphism $i : M' \to M$ and any homomorphism $g : M' \to I$ there exists $h : M \to I$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, tail, "i"] \arrow[d, "f"'] & M \arrow[dl, dashed, "h"] \\
 & I &
\end{tikzcd} \qquad *$$

(2) The functor $\operatorname{Hom}_A(-, I) : \underline{\underline{A-\mathrm{mod}}} \to \underline{\underline{A-\mathrm{mod}}}$ is exact
(3) Any exact sequence $0 \to I \to M \to M'' \to 0$ splits

Hint: to prove 3) implies 1), take any diagram as in 1) and consider the module

$$I \oplus_{M'} M \overset{def}{=} \frac{I \oplus M}{\{(g(m'), -i(m')) \,|\, m' \in M'\}},$$

called the push-out of the diagram *:

$$\begin{tikzcd}
M' \arrow[r, tail, "i"] \arrow[d, "f"'] & M \arrow[d] \\
I \arrow[r] & I \oplus_{M'} M
\end{tikzcd}$$

Show that the bottom horizontal map (induced by i) is injective; then apply 3) to that map.

Definition. A module satisfying one of these conditions is called **injective**.

In the next problem we shall describe injective modules over $\mathbb{Z}$. Note that this is a bit more involved than describing projective modules which are just $\mathbb{Z}^n$.

Problem 4.
I. Prove the **Baer's criterion** for injective modules: An A-module I is injective if and only if for any ideal $\mathfrak{a} \subset A$ and any map $f : \mathfrak{a} \to I$, the map f can be extended to $h : A \to I$:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r, tail] \arrow[d, "f"'] & A \arrow[dl, dashed, "h"] \\ & I & \end{tikzcd}$$

II. Show that an abelian group is injective (as a $\mathbb{Z}$-module) if an only if it is divisible. (An abelian group A is divisible if for any $a \in A$, and any $n \in \mathbb{Z}$ there exists $b \in A$ such that $a = nb$. For example, $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$ are divisible.)