Homework 1 for Group Cohomology, Winter 2006
due Monday, January 30

Unless specified otherwise, k is a field, G is a finite group.

Problem 0. Find as many (non-identical) explanations as you can to justify that $H^i(G, kG) = 0$ for $i > 0$.

Problem 1. Let $P(M)$ be the projective cover of a G-module M, S be a simple G-module, and $\Omega^i(M)$ be the i-th Heller shift of M.

(1) Show that $\mathrm{Hom}_G(P(M), S) = \mathrm{Hom}_G(M, S)$.
(2) Prove the "dimension shifting formula":

$$\mathrm{Ext}^i_G(M, S) = \mathrm{Hom}_G(\Omega^i(M), S)$$

(3) Assume that char k does not divide the order of G. Prove that $H^i(G, k) = 0$ for any $i > 0$.

Problem 2. Let M be a finite dimensional G module, and $M^{\#}$ be the linear dual of M, $M^{\#} = \mathrm{Hom}_k(M, k)$.
(a) Show that M is a direct summand of $M \otimes M^{\#} \otimes M$ as a G-module.
(b) Conclude that the following are equivalent:

(1) M is projective
(2) $\mathrm{End}_k(M, M) = M \otimes M^{\#}$ is projective
(3) $M^{\#}$ is projective
(4) M is injective

(c) Conclude that kG is a self-injective algebra, i.e. is an injective module over itself.
(d) Give an example of a ring R which is not an injective module over itself.

Problem 3. Compute normalized bar resolution of the trivial $\mathbb{Z}/2$-module $\mathbb{Z}$. Compare with the periodic resolution.

For the next two problems, denote by I the augmentation ideal of kG, i.e. $I = \mathrm{Ker}\,\{\epsilon : kG \to k\}$ where $\epsilon(g) = 1$ for any $g \in G$.
Problem 4. Show that

$$\mathrm{Ext}^1_G(k, k) \simeq \mathrm{Hom}_k(I/I^2, k).$$

Problem 5. Prove that $H_1(G, \mathbb{Z}) = G^{ab}$ using short exact sequence of G-modules $0 \to I \to \mathbb{Z}G \to \mathbb{Z} \to 0$.

Problem 6. Compute $H^i(\mathbb{Z}/p, \mathbb{Z}/p)$ and $H_i(\mathbb{Z}/p, \mathbb{Z}/p)$.

Problem 7. (Hilbert's theorem 90: multiplicative version). Let L/K be a finite Galois extension of fields, with Galois group G. Observe that G acts on the group of units of L, L^*.

Hilbert's theorem 90 states that if $\theta : G \to L^*$ is a derivation such that $\theta(gh) = g\theta(h)\theta(g)$, then $\theta(g) = (gx)/x$ for some $x \in L^*$.

(a) Show that *Hilbert's theorem 90* reformulates as

$$H^1(G, L^*) = 0.$$

(b) Show that $H^2(G, L^*)$ does not necessarily vanish.