MATH 341 Lin Alg I Nov. 11, 2003
Midterm **Name:** ________________________

No books, notes or calculators are allowed. Show ALL YOUR WORK.

(20) 1. Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix}$.

(a) Compute the product AB.

(b) Is it true or false that the product of two elementary matrices is again an elementary matrix?

Solution

(a) $AB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix}$.

(b) False. The matrices A and B above are both elementary, but their product AB is not.

(20) 2. Let $\{v_1, v_2, v_3\}$ be linearly independent vectors. Is it true or false that

(a) $\{v_1, v_2\}$ are linearly independent?

(b) $\{v_1 + v_2, v_2 + v_3\}$ are linearly independent?

Solution

(a) True. Proof by contradiction. Suppose v_1, v_2 are linearly dependent. Then, by definition of linear dependence, we can find scalars x, y, not both zero, such that $xv_1 + yv_2 = 0$. This implies that $xv_1 + yv_2 + 0v_3 = 0$. Applying definition once again, we conclude that v_1, v_2, v_3 are linearly dependent, which contradicts the initial assumption.

(b) True. Again, proof by contradiction. Suppose $v_1 + v_2, v_2 + v_3$ are linearly dependent. Then, by definition of linear dependence, we can find scalars x, y, not both zero, such that $x(v_1+v_2)+y(v_2+v_3) = 0$. This iplies that $xv_1+(x+y)v_2+yv_3 = x(v_1+v_2)+y(v_2+v_3) = 0$. Thus, we found a non-trivial linear combination of v_1, v_2, v_3 equal to zero. Applying definition once again, we conclude that v_1, v_2, v_3 are linearly dependent, which contradicts the initial assumption.

(20) 3. Let $A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 5 & 0 & 1 \end{pmatrix}$.

(a) Find A^{-1} if A is invertible. What is the dimension of the column space of A?

(b) Let $b = \begin{pmatrix} 2 \\ 7 \\ 201 \end{pmatrix}$. Is the system $Ax = b$ consistent? If so, how many solutions does it have?

Answers

(a) A is not invertible, the REF of A has pivots in columns 1 and 2. Thus, the dimension of the column space is 2.

(b) The system is inconsistent, it has 0 solutions.

(20) 4. Do the planes $x - 2y + z = 0$, $2y - 8z = 8$, and $-4x + 5y - 9z = -9$ in $\mathbf{R}^3$ intersect? Find all points in the intersection.

Answer

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6\frac{13}{17} \\ 3\frac{5}{17} \\ -\frac{3}{17} \end{bmatrix} = \begin{bmatrix} \frac{115}{17} \\ \frac{56}{17} \\ -\frac{3}{17} \end{bmatrix}$$

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(20) 5. Let $A = \begin{pmatrix} 1 & 2 & 3 & -1 \\ 5 & -4 & 0 & -2 \\ 3 & -8 & -6 & 4 \\ 7 & 0 & 6 & 0 \end{pmatrix}$

(a) Compute Det A.

(b) Compute Rank A.

(c) Find basis for the column space of A.

(d) Find basis for the Null space of A. What is the dimension of the null space of A (= nullity of A)?

Answers

(a) Det A = 0

(b) Rank A = 3

(c) Pivoting columns are 1, 2 and 4.

Basis of the column space: $\begin{bmatrix} 1 \\ 5 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 4 \\ 0 \end{bmatrix}$

(d) Basis of the null space: $\begin{bmatrix} -\frac{6}{7} \\ -\frac{15}{14} \\ 1 \\ 0 \end{bmatrix}$. Nullity = 1.

(5) 6. (*Bonus.*) Find the area of the ellipse given by the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$. (Hint: observe that the ellipse is the image of a unit circle under the linear transformation given by dilations along x and y-axes)

Solution

Vertices of the ellipse have coordinates $(2, 0)$, $(-2, 0)$, $(0, 3)$ and $(0, -3)$. The linear transformation that takes the unit circle to this ellipse is given by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. The determinant of this matrix is 6. Thus, the area of the unit circle, which is π, is muplitplied by 6, and we get 6π for the area of the ellipse.