MATH 341 Linear Algebra I Fall 2003
Final Exam **Name:** ____________________

No books, notes or calculators are allowed. Show ALL your work.

(30) 1. Mark each of the following statements true or false. You don't have to give an explanation.

(a) $\det AB = \det A \cdot \det B$

(b) $\det (A + B) = \det A + \det B$

(c) $\det A = \det A^{-1}$

(d) If A is invertible, then A^T is invertible

(e) If A is an invertible square matrix of dimension $n \times n$, then $\operatorname{rk} A = n$.

(f) $AB = BA$

(g) If $\det A \neq \det B$, then A and B cannot be row equivalent

(h) $\operatorname{rk} A^T = \operatorname{rk} A$

(i) If A is an invertible matrix $n \times n$, then $Ax = b$ is consistent for any $b \in \mathbf{R}^n$

(j) If a vector space V has dimension n, then any set of n vectors in V form a basis

(25) 2. (a) Check that vectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$$

form a basis of $\mathbf{R^3}$.

(b) Find coordinates of the vector $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ with respect to the basis above.

(20) 3. Compute the volume of a parallelepiped with vertices $(1, 0, 1)$, $(2, 1, 2)$, $(2, 2, 5)$, $(2, -3, 10)$, $(3, 3, 6)$, $(3, -1, 14)$, $(3, -2, 11)$, $(4, 0, 15)$.

(25) 4. Let $T : \mathbf{R^2} \to \mathbf{R^2}$ be a linear transformation.

(a) Define what it means for T to be *onto*.

(b) Suppose the kernel of T is a line. Can T be onto? Explain.

(c) Define what it means for T to be *one-to-one*.

(d) Suppose T is a projection onto the x-axis. Is T one-to-one? Describe $Ker\, T$ geometrically.

(e) Let T be a rotation by 90° counterclockwise. Write the matrix of T. Explain why T is onto and one-to-one.

(30) 5. (a) Define basis and dimension of a vector space V.

(b) Determine whether the set of vectors $1+t+t^3, 2t^2-t^3, 3-2t-t^2+2t^3, 4+4t+t^2+3t^3$ forms a basis of the vector space P_3, i.e. the space of all real polynomials of degree at most 3. State the dimension of this vector space.

(25) 6. Let A be a matrix of dimension $m \times n$. Finish the equalities

(a) $\operatorname{rk} A + \dim \operatorname{Nul} A =$

(b) $\operatorname{rk} A + \dim \operatorname{Nul} A^T =$

(c) Using (a) and (b), show that a square matrix A is invertible if and only if A^T is invertible.

(25) 7. (a) Let $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Using Cramer's rule, solve matrix equations $Ax_1 = e_1$, $Ax_2 = e_2$.

(b) What is A^{-1}?

(c) Let A be an arbitrary invertible matrix 2×2 and e_1, e_2 be as in (a). Let x_1, x_2 be the solutions of the matrix quations $Ax_1 = e_1$ and $Ax_2 = e_2$. Explain why x_1, x_2 are linearly independent.

(25) 8. (a) Define kernel of a linear transformation $T : V \to W$.

(b) Show that Ker T is always a subspace of V.

(b) Let T be a linear transformation of $\mathbf{R^3}$ given by the matrix $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 3 & -5 \\ 1 & 2 & -3 \end{bmatrix}$.

Find basis and dimension of Ker T.

(5) 9. (**Bonus.**) A linear transformation $T : R^n \to R^n$ is called area preserving if for any shape $S \subset R^n$, area of $T(S)$ is the same as area of S. Let A be the matix of the linear transformation T. Give a numerical condition on A which would guarantee that T is area preserving.