**MATH 341 Lin Alg I Fall 2003**
**Final Exam** **Name:** ____________________

No books, notes or calculators are allowed. Show ALL your work.

(30) 1. Mark each of the following statements true or false. You don't have to give an explanation.

(a) $\det AB = \det A \cdot \det B$

(b) $\det(A+B) = \det A + \det B$

(c) $\det A = \det A^{-1}$

(d) If $A$ is invertible, then $A^T$ is invertible

(e) If $A$ is an invertible square matrix of dimension $n \times n$, then $\operatorname{rk} A = n$.

(f) $AB = BA$

(g) If $\det A \neq \det B$, then $A$ and $B$ cannot be row equivalent

(h) $\operatorname{rk} A^T = \operatorname{rk} A$

(i) If $A$ is an invertible matrix $n \times n$, then $Ax = b$ is consistent for any $b \in \mathbf{R}^n$

(j) If a vector space $V$ has dimension $n$, then any set of $n$ vectors in $V$ form a basis

**Solution.**

(a) True

(b) False

(c) False

(d) True

(e) True

(f) False

(g) True

(h) True

(i) True

(j) False

(25) 2. (a) Check that vectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$$

form a basis of $\mathbf{R^3}$.

(b) Find coordinates of the vector $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ with respect to the basis above.

**Solution.**

(a) Compute determinant of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 7 \end{bmatrix}$ by decomposing with respect to the second row. We get $\det A = 13$. Since $\det A \neq 0$, the matrix is invertible and its columns are linearly independent. Thus, they form a basis.

(b) Solve the system

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 7 & 6 \end{array}\right]$$

by row reduction. We get $(1, 1/13, 11/13)$.

(20) 3. Compute the volume of a parallelepiped with vertices $(1,0,1)$, $(2,1,2)$, $(2,2,5)$, $(2,-3,10)$, $(3,3,6)$, $(3,-1,14)$, $(3,-2,11)$, $(4,0,15)$.

**Solution.**

Shifting by the vector $(1,0,1)$, we get a parallelepiped with one vertex at the origin, based on vectors $(1,1,1)$, $(1,2,4)$, and $(1,-3,9)$.

$$\text{Volume } = \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = 20$$

(25) 4. Let $T : \mathbf{R^2} \to \mathbf{R^2}$ be a linear transformation.

(a) Define what it means for $T$ to be *onto*.

(b) Suppose the kernel of $T$ is a line. Can $T$ be onto? Explain.

(c) Define what it means for $T$ to be *one-to-one*.

(d) Suppose $T$ is a projection onto the $x$-axis. Is $T$ one-to-one? Describe $Ker\, T$ geometrically.

(e) Let $T$ be a rotation by 90° counterclockwise. Write the matrix of $T$. Explain why $T$ is onto and one-to-one.

**Solution.**

(b) No. By the Rank Theorem, dim Ker $T$ + dim Im $T = 2$. Since dim Ker $T = 1$, we conclude that dim Im $T = 1$. But for $T$ to be onto, image of $T$ has to be the entire $R^2$ which has dimension 2.

(d) No. Projection is not one-to-one, because all vectors of the form $(0, y)$ map to zero. Kernel of $T$ is the $y$-axis, which is a line.

(e) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Matrix is invertible, therefore, $T$ is onto and one-to-one.

(30) 5. (a) Define basis and dimension of a vector space $V$.

(b) Determine whether the set of vectors $1+t+t^3, 2t^2-t^3, 3-2t-t^2+2t^3, 4+4t+t^2+3t^3$ forms a basis of the vector space $P_3$, i.e. the space of all real polynomials of degree at most 3. State the dimension of this vector space.

**Solution.**

(b) The standard basis of $P_3$ is given by polynomials $(1, t, t^2, t^3)$. With respect to the standard basis, new vectors have coordinates $(1, 1, 0, 1)$, $(0, 0, 2, -1)$, $(3, -2, -1, 2)$, and $(4, 4, 1, 3)$. We compute the determinant of the matrix with corresponding columns:

$$\det \begin{bmatrix} 1 & 0 & 3 & 4 \\ 1 & 0 & -2 & 4 \\ 0 & 2 & -1 & 1 \\ 1 & -1 & 2 & 3 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 0 & -5 & 0 \\ 0 & 2 & -1 & 1 \\ 1 & -1 & 2 & 3 \end{bmatrix} = -5 \cdot \det \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 1 & -1 & 2 & 3 \end{bmatrix} = -5 \cdot$$

$$\det \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix} = 5 \cdot \det \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = 5 \cdot \det \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix} = 5 \cdot$$

$$\det \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & -1 \end{bmatrix} = 5 \cdot \det \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} == -5.$$

Since $\det \neq 0$, vectors are linearly independent. The dimension of $P_3$ is 4, because the standard basis has 4 elements. Since the new set consists of 4 linearly independent vectors, the Basis theorem implies that they form a basis.

(25) 6. Let $A$ be a matrix of dimension $m \times n$. Finish the equalities

(a) $\operatorname{rk} A + \dim \operatorname{Nul} A =$

(b) $\operatorname{rk} A + \dim \operatorname{Nul} A^T =$

(c) Using (a) and (b), show that a square matrix $A$ is invertible if and only if $A^T$ is invertible.

**Solution.**

(a) $\operatorname{rk} A + \dim \operatorname{Nul} A = n$

(b) $\operatorname{rk} A + \dim \operatorname{Nul} A^T = m$

(c) Let $A$ be $n \times n$. In this case, (a) + (b) implies that $\dim \operatorname{Nul} A = \dim \operatorname{Nul} A^T$. $A$ is invertible if and only if $Ax = 0$ does not have non-trivial solutions if and only if $\dim \operatorname{Nul} A = 0$ if and only if $\dim \operatorname{Nul} A^T = 0$ if and only if $A^T x = 0$ does not have non-trivial solutions if and only if $A^T$ is invertible.

(25) 7. (a) Let $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Using Cramer's rule, solve matrix equations $Ax_1 = e_1$, $Ax_2 = e_2$.

(b) What is $A^{-1}$?

(c) Let $A$ be an arbitrary invertible matrix $2 \times 2$ and $e_1, e_2$ be as in (a). Let $x_1, x_2$ be the solutions of the matrix quations $Ax_1 = e_1$ and $Ax_2 = e_2$. Explain why $x_1, x_2$ are linearly independent.

**Solution.**

(a) $\det A = 1$.

For $Ax_1 = e_1$, $\det A_1(e_1) = -1$, $\det A_2(e_2) = -2$.

For $Ax_2 = e_2$, $\det A_1(e_2) = 1$, $\det A_2(e_2) = 1$.

Applying Cramer's formula, we obtain

$$x_1 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) $A^{-1} = [x_1\ x_2] = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$

(c) Observe that $A \cdot [x_1\ x_2] = [e_1\ e_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, $A^{-1} = [x_1\ x_2]$. As columns of the invertible matrix $A^{-1}$, $x_1, x_2$ must be linearly independent.

Alternatively, one may observe that if $x_1, x_2$ satisfy some non-trivial linear relation, i.e. $ax_1 + bx_2 = 0$, then $e_1, e_2$ satisfy the same linear relation since $ae_1 + be_2 = aAx_1 + bAx_2 = A(ax_1 + bx_2) = A(0) = 0$. But this is impossible, since $e_1, e_2$ form the standard basis in $\mathbf{R}^2$ and, in particular, are linearly independent.

(25) 8. (a) Define kernel of a linear transformation $T : V \to W$.

(b) Show that Ker $T$ is always a subspace of $V$.

(c) Let $T$ be a linear transformation of $\mathbf{R^3}$ given by the matrix $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 3 & -5 \\ 1 & 2 & -3 \end{bmatrix}$.

Find basis and dimension of Ker $T$.

**Solution.**

(a) $Ker\, T = \{x \in V : T(x) = 0\}$.

(b) If $x, y \in Ker\, T$, then $T(x) = T(y) = 0$. Therefore, $T(x+y) = T(x)+T(y) = 0+0 = 0$. Thus, $x + y \in Ker\, T$. We conclude that $Ker\, T$ is closed under addition. Similarly, if $T(x) = 0$, then $T(cx) = cT(x) = 0$ for any real number $c$. Thus, $Ker\, T$ is closed under scalar mutiplication. Now it is a subspace by definition.

(c) $Ker\, T = Nul\, A$. To find basis of $Nul\, A$, we solve homogeneous system $Ax = 0$. The solutions are $z \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. Thus, $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ is a basis of $Nul\, A$, and dimension is 1.