MATH 341 Lin Alg I Fall 2003
Practice problems for final exam

1. Let A be a matrix.

   (a) Define Rk A.

   (b) Is it true that Rk A = Rk A^T? Justify your answer.

   (c) Is it true that dim Nul A = dim Nul A^T? Justify your answer.

Solution.

(a) $\mathrm{rk}\, A$ is the dimension of the column space of A.

(b) True. $\mathrm{rk}\, A = \dim\ \mathrm{Col}\, A$ = number of pivots in the reduced echelon form of A = dimension of the row space of $A = \dim\ \mathrm{Col}\, A^T = \mathrm{rk}\, A^T$

(c) False. Let A be a 2×3 matrix of rank 2 (i.e. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$). Then Dim Nul A = 3 - rk A = 1 (there is 1 free variable), but dim Nul A^T = 2 - rk A^T = 2 - 2 = 0 (in $A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ there are no free variables).

2. Let S be a parallelogram determined by the vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and T be a linear transformation determined by the matrix $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$. Compute the area of $T(S)$.

Solution.

$$\text{Area of } T(S) = \det A \bullet \text{Area of } S = \det \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \bullet \det \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = 2 \bullet 3 = 6$$

3. Compute the solutions of the system using Cramer's rule:

$$\begin{cases} 2x_1 + x_2 = 7 \\ -3x_1 + x_3 = -8 \\ x_2 + 2x_3 = -3 \end{cases}$$

Solution.
Det $A = 4$
Det $A_1(b) = 6$
Det $A_2(b) = 16$
Det $A_3(b) =$ -14
Solution: $(3/2, 4, -7/2)$.

4. Give definition of the dimension of a vector space.

Determine whether the following sets are vector spaces (justify your answers). If the are, state their dimensions.

(a) All real polynomials of degree at most 5.

(b) All real polynomials of degree exactly 5.

(c) All real polynomials of degree at most two with constant coefficient 1 (i.e. $x+1$, x^2+2x+1, etc.)

(d) The set of all x such that $Ax = b$ where $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 3 & -5 \\ 1 & 2 & -3 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}$.

Solution

A vector space is called *finite-dimensional* if it is spanned by a finite set of vectors. In that case, dimension of the vector space is the number of vectors in a basis. Otherwise, vectors space is called *infinite-dimensional*.

(a) Yes. Dim = 6.

(b) No. Does not contain 0-polynomial.

(c) No. Does not contain 0-polynomial.

(d) No. 0-vector ($\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$) is not a solution.

5. Let P_3^o be a vector space of real polynomials of degree at most 3 with zero constant term. Using the fact that t, t^2, t^3 form a basis of P_3^o, show that t^2+t, t^3-t and $2t^2+t^3$ form a basis as well. Compute the coordinate vector of t^3+t^2+t with respect to the latter basis.

Solution.

Let $< t, t^2, t^3 >$ be the standard basis of P_3^o. The change-of-cordinate matrix, $B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, is invertible because $\det B = -1$. Therefore, t^2+t, t^3-t and $2t^2+t^3$ are linearly independent and, thus, form a basis.

The coordinates of t^3+t^2+t in the standard basis are $(1,1,1)$. To find the coordinates of t^3+t^2+t with respect to the new basis, we have to solve a linear system with the augmented matrix

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

The solution is $(3, 2, -1)$.

6. Let

$$s\begin{bmatrix}2\\-2\\3\end{bmatrix}+t\begin{bmatrix}4\\-5\\7\end{bmatrix}$$

and

$$x\begin{bmatrix}-2\\7\\-8\end{bmatrix}+y\begin{bmatrix}1\\3\\6\end{bmatrix}$$

be two planes in $\mathbf{R^3}$ given parametrically. Describe their intersetion.

Solution.

Intersection consists of all vectors which can be written both as $s\begin{bmatrix}2\\-2\\3\end{bmatrix}+t\begin{bmatrix}4\\-5\\7\end{bmatrix}$ and $x\begin{bmatrix}-2\\7\\-8\end{bmatrix}+y\begin{bmatrix}1\\3\\6\end{bmatrix}$. To find such vectors, we have to solve homogeneous system of equations

$$s\begin{bmatrix}2\\-2\\3\end{bmatrix}+t\begin{bmatrix}4\\-5\\7\end{bmatrix}=x\begin{bmatrix}-2\\7\\-8\end{bmatrix}+y\begin{bmatrix}1\\3\\6\end{bmatrix}.$$

Transferring everything to the left hand side, we get a homogeneous system $Ax=0$ where

$$A=\begin{bmatrix}2&4&2&-1\\-2&-5&-7&-3\\3&7&8&-6\end{bmatrix}.$$

Solving by row reduction, we get a line of solutions given parametrically by

$$\begin{bmatrix}s\\t\\x\\y\end{bmatrix}=u\begin{bmatrix}-9\\5\\1\\0\end{bmatrix},$$

where u is a free variable. Therefore, the intersection is a line, and it consists of all vectors of the form

$$-9u\begin{bmatrix}2\\-2\\3\end{bmatrix}+5u\begin{bmatrix}4\\-5\\7\end{bmatrix}$$

or, equivalently,

$$u\begin{bmatrix}-2\\7\\-8\end{bmatrix}.$$

7. (a) Define the image of a linear transformation $T : V \to W$.

(b) Show that Im T is always a subspace of W.

(b) Let T be a linear transformation of $\mathbf{R^3}$ given by a matrix $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 3 & -5 \\ 1 & 2 & -3 \end{bmatrix}$. Find basis and dimension of Im T.

Solution.

(a) Im $T = \{w \in W, \text{such that } T(v) = w \text{ for some } v \in V\}$.

(b) This was a homework problem.

(c) In this case, Im T = Col A. To find basis and dimension of Col A, we row reduce A to a matrix in the echelon form,

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 3 & -5 \\ 1 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

observe that the pivoting columns are columns 1 and 2, and conclude that a basis of the column space is given by the first two columns of the matrix A, and the dimension is 2.